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ELEVEN-DIMENSIONAL SUPERGRAVITY ON A MANIFOLD WITH BOUNDARY

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In this paper, we present a systematic analysis of eleven-dimensional supergravity on a manifold with boundary, which is believed to be relevant to the strong coupling limit of the $E_8 \times E_8$ heterotic string. Gauge and gravitational anomalies enter at a very early stage, and require a refinement of the standard Green-Schwarz mechanism for their cancellation. This uniquely determines the gauge group to be a copy of E_8 for each boundary component, fixes the gauge coupling constant in terms of the gravitational constant, and leads to several striking new tests of the hypothesis that there is a consistent quantum M-theory with eleven-dimensional supergravity as its low energy limit.

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1. Introduction

In a previous paper [1], we proposed that the strong coupling limit of the ten-dimensional $E_8 \times E_8$ heterotic string is eleven-dimensional M-theory compactified on $\mathbf{R}^{10} \times \mathbf{S}^1/\mathbf{Z}_2 = \mathbf{R}^{10} \times I$ (I is the unit interval), with the gauge fields entering via ten-dimensional vector multiplets that propagate on the boundary of space-time. This implies in particular that there must exist a supersymmetric coupling of ten-dimensional vector multiplets on the boundary of an eleven-manifold to the eleven-dimensional supergravity multiplet propagating in the bulk. The purpose of the present paper is to explore this coupling.

In doing so, one comes quickly to a puzzle. The supergravity action in bulk is

$$-\frac{1}{2\kappa^2}\int_{M^{11}} d^{11}x\sqrt{g}\,R + \dots, \tag{1.1}$$

with M^{11} being the eleven-dimensional space-time, and "..." being the terms involving fermions and the bosonic three-form field. The supergauge action on the boundary is

$$-\frac{1}{4\lambda^2}\int_{M^{10}} d^{10}x\sqrt{g}\,\mathrm{tr}\,F^2 + \dots, \tag{1.2}$$

where M^{10} is the boundary (or a component of the boundary) of M^{11}, and F is the field strength of the gauge fields that propagate on M^{10}. (For $E_8 \times E_8$, "tr" is as usual $1/30$ of the trace Tr in the adjoint representation.) In the above formulas, κ and λ are the gravitational and gauge coupling constants. From those constants one can make a dimensionless number $\eta = \lambda^6/\kappa^4$. The question is what determines the value of η. Note that there is no dilaton or other scalar whose expectation value controls the value of η. In fact, there is no scalar field at all in the theory, propagating either in bulk or on the boundary; in going to strong coupling, the dilaton of the perturbative heterotic string is reinterpreted as the radius of $\mathbf{S}^1/\mathbf{Z}_2$.

Since string theory has no adjustable parameter corresponding to η, the strong coupling limit of the $E_8 \times E_8$ heterotic string, if it does have the eleven-dimensional interpretation proposed in our previous paper, must give a definite value for η. In fact, we will argue in this paper that by looking more precisely at gravitational and gauge anomalies (which were already used in the previous paper), one can determine η. We get

$$\eta = 128\pi^5, \tag{1.3}$$

or equivalently

$$\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}. \tag{1.4}$$

In the remainder of this introduction, we sketch the argument that will be used to determine η, and also sketch the other main qualitative results of this paper. The reason for presenting such a detailed sketch first is that the supergravity calculation that occupies the remainder of the paper is unavoidably rather complicated.

We recall that anomalies in ten dimensions are described by a formal twelve-form $I_{12}(R, F_1, F_2)$ that is a sixth order homogeneous polynomial in the Riemann tensor R and the field strengths F_1 and F_2 in the two E_8's. It has the general form

$$I_{12}(R, F_1, F_2) = A(R) + B(R, F_1) + B(R, F_2), \tag{1.5}$$

where $A(R)$ is the contribution of the supergravity multiplet, and $B(R, F_i)$, for $i = 1, 2$, is the contribution of the gluinos of the i^{th} E_8. In [1], we introduced

$$\widehat{I}_{12}(R, F) = \frac{1}{2}A(R) + B(R, F), \tag{1.6}$$

so that

$$I_{12}(R, F_1, F_2) = \widehat{I}_{12}(R, F_1) + \widehat{I}_{12}(R, F_2). \tag{1.7}$$

The idea here is that from the eleven-dimensional point of view, the gauge and gravitational anomaly is localized on the boundary of space-time, and the two terms on the right of (1.7) are the contributions of the two components of the boundary of $\mathbf{R}^{10} \times I$. Of the two E_8's, the one propagating on a given boundary component is naturally the only one that contributes to the anomaly form of that component.

Anomaly cancellation for the perturbative heterotic string involves a factorization

$$I_{12} = I_4 I_8, \tag{1.8}$$

where $I_4 = \mathrm{tr}\, R^2 - \mathrm{tr}\, F_1^2 - \mathrm{tr}\, F_2^2$ and I_8 is an eight-form given by a lengthy quartic polynomial in R and the F_i. As was explained in [1], $\widehat{I}_{12}$ has an analogous factorization

$$\widehat{I}_{12} = \widehat{I}_4 \widehat{I}_8, \tag{1.9}$$

with

$$\begin{aligned} \widehat{I}_4(R, F) =& \frac{1}{2}\mathrm{tr}\, R^2 - \mathrm{tr}\, F^2 \\ \widehat{I}_8(R, F) =& -\frac{1}{4}\widehat{I}_4(R, F)^2 + \left(-\frac{1}{8}\mathrm{tr}\, R^4 + \frac{1}{32}(\mathrm{tr}\, R^2)^2\right). \end{aligned} \tag{1.10}$$

(This way of writing the formula for $\widehat{I}_8$, which was also noticed by M. Duff and R. Minasian, has a rationale that will become clear in section three.) It was proposed in [1] that this factorization of $\widehat{I}_{12}$ would permit an extension of the Green-Schwarz anomaly cancellation mechanism to M-theory on eleven-dimensional manifolds with boundary.

The Green-Schwarz mechanism in ten dimensions depends on the existence in string theory of a two-form field B whose gauge-invariant field strength H obeys

$$dH = I_4. \tag{1.11}$$

Such an equation (with only the $\mathrm{tr}\, F^2$ term in I_4) holds even in the minimal ten-dimensional supergravity [2,3]. In addition, there are "Green-Schwarz interaction terms," present in the string theory but *not* in the minimal low energy supergravity theory, of the form

$$\Delta L = \int B \wedge I_8. \tag{1.12}$$

The combination of (1.11) and (1.12) gives a classical theory that is *not* gauge invariant, with an anomaly constructed from the twelve-form $I_{12} = I_4 I_8$. The minimal classical supergravity theory is gauge invariant because the anomalous fermion loops and the Green-Schwarz terms are both absent, and the string theory is gauge invariant because they are both present and the anomalies cancel between them.

Now let us discuss how the story will work in eleven dimensions. In doing so, and in most of this paper, we will use an orbifold approach in which we work on an eleven-manifold M^{11} with a $\mathbf{Z}_2$ symmetry whose fixed points are of codimension one; alternatively, one can take the quotient and work on the manifold-with-boundary $X = M^{11}/\mathbf{Z}_2$, whose boundary points are the $\mathbf{Z}_2$ fixed points in M^{11}. In general, the formulation in terms of a manifold with boundary is convenient intuitively, and the orbifold formulation is convenient for calculation.

Rather than a two-form B, the eleven-dimensional supergravity multiplet has a three-form field C (denoted by $A^{(3)}$ in our previous paper [1]), whose field strength is a four-form G.[1] In the absence of boundaries, G obeys the usual Bianchi identity $dG = 0$. The analog of (1.11) will have to be a contribution to dG supported at the $\mathbf{Z}_2$ fixed points. As dG is a five-form, we will have to promote the four-form $\widehat{I}_4$ to a five-form supported on the fixed

[1] Following conventions in [2] which have become standard in eleven-dimensional supergravity, we define $G_{IJKL} = \partial_I C_{JKL} \pm 23$ terms, though the normalization is somewhat unusual. We also define $dG_{IJKLM} = \partial_I G_{JKLM} +$ cyclic permutations of $IJKLM$.

point set, so that it can appear as a correction to the Bianchi identity. To write such a five-form, one supposes that the fixed point set is defined locally by an equation $x^{11} = 0$, and one multiplies by the closed one-form $\delta(x^{11})dx^{11}$ to promote the four-form $\widehat{I}$ to a five-form. Thus, the eleven-dimensional analog of the ten-dimensional equation $dH = I_4$ will be an equation $dG = c\delta(x^{11})dx^{11}\widehat{I}_4$, with some constant c. In section two, we will determine the precise equation to be

$$dG_{11\,IJKL} = -\frac{3\sqrt{2}}{2\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta(x^{11})\left(\mathrm{tr}\,F_{[IJ}F_{KL]} - \frac{1}{2}\mathrm{tr}\,R_{[IJ}R_{KL]}\right). \tag{1.13}$$

Here F is of course the field strength of the gauge field propagating at $x^{11} = 0$, and $\mathrm{tr}\,F_{[IJ}F_{KL]} = (1/24)\mathrm{tr}\,F_{IJ}F_{KL} \pm$ permutations. Actually, in section two, we will see directly only the $\mathrm{tr}\,F\wedge F$ term in (1.13); the $\mathrm{tr}\,R\wedge R$ term is a sort of higher order correction that we infer because it is needed for anomaly cancellation. (Analogously, in ten dimensions, the $\mathrm{tr}\,F\wedge F$ term is required by supersymmetry, and the $\mathrm{tr}\,R\wedge R$ term is an $O(\alpha')$ stringy correction needed for anomaly cancellation.)

At this stage the question is, what are the Green-Schwarz terms? In the familiar ten-dimensional story, because the Green-Schwarz terms are unconstrained by supersymmetry, the Green-Schwarz mechanism makes no general prediction (independent of anomalies or a detailed string model) about what I_8 should be. In eleven dimensions, the story will be quite different because the terms analogous to the Green-Schwarz terms are independently known. One of these terms is simply the $\int C\wedge G\wedge G$ interaction of eleven-dimensional supergravity. This term, discovered when the model was first constructed [8], has always seemed enigmatic because the rationale behind its apparently "topological" nature was not clear. We feel that the role of this term in canceling anomalies – we explain in section three how $\int C\wedge G\wedge G$ comes to play the role of a Green-Schwarz term – removes some of the enigma.

The $\int C\wedge G\wedge G$ is the only "Green-Schwarz" interaction involved in canceling gauge anomalies, but to cancel also the gravitational anomalies requires an additional interaction. This is an eleven-dimensional interaction

$$\int_{M^{11}} C\wedge X_8(R), \tag{1.14}$$

with $X_8(R)$ an eight-form constructed as a quartic polynomial in the Riemann tensor. This interaction is known in two ways. (1) Upon dimensional reduction on $\mathbf{S}^1$, it turns into a $B\wedge X_8$ interaction which can be computed as a one-loop effect in Type IIA superstring

theory [7]. The one loop calculation is exact since a dilaton dependence of the $B \wedge X_8$ coupling would spoil gauge invariance; because it is exact, it can be extrapolated to eleven dimensions and implies the existence of the interaction written in (1.14). (2) Alternatively, this coupling is needed to cancel one-loop anomalies on the five-brane world-volume and thus permit the existence of five-branes in the theory [4-6]. Happily, the two methods agree, with

$$X_8 = -\frac{1}{8}\mathrm{tr}\, R^4 + \frac{1}{32}(\mathrm{tr}\, R^2)^2. \tag{1.15}$$

As we will see in section three, it is no coincidence that the combination of $\mathrm{tr}\, R^4$ and $(\mathrm{tr}\, R^2)^2$ that appears here also entered in (1.10).

The fact that the terms analogous to Green-Schwarz terms are known independently of any discussion of space-time anomalies means that we get an *a priori* prediction for $\widehat{I}_8$. (We have no *a priori* prediction of $\widehat{I}_4$, as the coefficients in (1.13) will essentially be adjusted to make anomaly cancellation possible.) We regard the success of this prediction as a compelling confirmation that eleven-dimensional supergravity on a manifold with boundary is indeed related to ten-dimensional $E_8 \times E_8$ heterotic string theory as proposed in [1].

Classical and Quantum Consistency

The details that we have just explained of how anomaly cancellation works in eleven dimensions have other implications for the structure of the theory.

The fact that, once one works on a manifold with boundary, some of the Green-Schwarz terms are present in the minimal supergravity Lagrangian means that the classical Lagrangian, including the vector supermultiplets on the boundary, is not gauge invariant. Thus, the theory with the supergravity multiplet in bulk and the vector multiplets on the boundary is only consistent as a quantum theory. The situation is rather different from perturbative string theory, where since the Green-Schwarz terms arise at the one loop level, one has gauge invariance either classically (leaving out the anomalous chiral fermion loop diagrams and the effects of the Green-Schwarz terms) or quantum mechanically (including both of these).

The relation $\lambda^2 \sim \kappa^{4/3}$ between the gauge and gravitational couplings sheds a further light on this. It means that the gauge kinetic energy, of order $1/\lambda^2$, is a higher order correction, of relative order $\kappa^{2/3}$ compared to the gravitational action, which is of order $1/\kappa^2$. If one wants a fully consistent classical theory, one must ignore the gauge fields completely. Once one tries to include the supergauge multiplet, gauge invariance will fail

classically (in relative order κ^2), and quantum anomalies are needed to compensate for this failure.

Since the classical theory with the gauge fields is not going to be fully consistent, one has to expect peculiarities in constructing it. In our analysis in section four, we certainly find such peculiarities. We will organize our discussion of the boundary interactions as an expansion in powers of $\kappa^{2/3}$. In order $\kappa^{2/3}$, things go smoothly, though the calculations are rather involved, roughly as in standard supergravity theories. Some novelties arise in order $\kappa^{4/3}$. In verifying invariance of the Lagrangian in that order, one has to cancel terms that are formally proportional to $\delta(0)$. The cancellation also involves adding to the Lagrangian new interactions (of relative order $\kappa^{4/3}$) proportional to $\delta(0)$. We interpret the occurrence of $\delta(0)$ terms in the Lagrangian and the supersymmetry variations of fields as a symptom of attempting to treat in classical supergravity what really should be treated in quantum M-theory. In a proper quantum M-theory treatment, there would presumably be a built-in cutoff that would replace $\delta(0)$ by a finite constant times $\kappa^{-2/9}$. For instance, the cutoff might involve having the gauge fields propagate in a boundary layer, with a thickness of order $\kappa^{2/9}$, and not precisely on the boundary of space-time.

Though the $\delta(0)$ terms formally cancel in order $\kappa^{4/3}$, one must expect further difficulties in higher order since, without knowing the correct way to cut off the linear divergence that gave the $\delta(0)$ terms in order $\kappa^{4/3}$, there is some uncertainty in the determination of the correct structure in that order. One must suppose, by analogy with many other problems in physics, that underneath the cancellation of the linear divergences there might be a finite remainder, which could be extracted if one understood the correct cutoff. Without understanding the finite remainder, one should expect difficulty in proceeding to the next order.

In any event, one of the things that happens in the next order – relative order κ^2 – has already been explained. One runs into a failure of classical gauge invariance which must be canceled by quantum one-loop anomalies (which are also of relative order κ^2). It is hard to believe that the classical discussion can usefully be continued to higher order, once the classical gauge invariance has failed and one has begun to run into conventional quantum loops. An attempt to continue the classical discussion would almost undoubtedly soon run into higher order divergences than the $\delta(0)$ that we described two paragraphs ago; for instance, one would very likely find $\delta(0)$ terms in the supergravity transformation laws and $\delta(0)^2$ terms in the Lagrangian.

Despite the infinities that arise in the construction, we hope and expect that the analysis of the anomalies is reliable. This should be analogous to the fact that anomalous loop diagrams can be reliably computed even in unrenormalizable effective theories, because the anomalies can be construed as an infrared effect and are independent of what cutoff one introduces.

Summary

To summarize, then, the lessons from our investigation, we will find that anomaly cancellation of the ten-dimensional heterotic string has an elegant eleven-dimensional interpretation that sheds light on properties of the anomaly twelve-form that were not needed before. This sharpens the eleven-dimensional interpretation of the strongly coupled $E_8\times E_8$ heterotic string, fixing an otherwise unknown dimensionless parameter and adding to our confidence that the eleven-dimensional description is correct. The gauge anomalies that arise in the classical discussion also give an indication – and not the only one – that the theory only really makes sense at the quantum level.

2. Correction to the Bianchi Identity

Our eleven-dimensional conventions are as in [2]. We work with Lorentz signature $-+++\ldots+$. Vector indices will be written as I,J,K, and spinor indices as α,β,γ. We introduce a frame field $e_I{}^m$ with the metric being $g_{IJ}=\eta_{mn}e_I{}^m e_J{}^n$. The gamma matrices are 32×32 real matrices obeying $\{\Gamma_I,\Gamma_J\}=2g_{IJ}$. One also defines $\Gamma^{I_1I_2\ldots I_n}=\Gamma^{[I_1}\ldots\Gamma^{I_n]}\equiv(1/n!)\Gamma^{I_1}\Gamma^{I_2}\ldots\Gamma^{I_n}\pm$ permutations. Spinor indices are raised and lowered with a real antisymmetric tensor $\mathcal{C}$ obeying $\mathcal{C}_{\alpha\beta}=-\mathcal{C}_{\beta\alpha}$, $\mathcal{C}^{\alpha\beta}\mathcal{C}_{\beta\gamma}=\delta^\alpha_\gamma$. In particular, by lowering an index in the gamma matrix $\Gamma^\alpha_{I\,\beta}$ one gets a symmetric tensor $\Gamma_{I\alpha\beta}=\Gamma_{I\beta\alpha}$. All spinors will be Majorana spinors; the symbol $\overline{\psi}_\alpha$ is simply defined by $\overline{\psi}_\alpha=\mathcal{C}_{\alpha\beta}\psi^\beta$.

The supergravity multiplet consists of the metric g, the gravitino $\psi_{I\alpha}$, and a three-form C (with field strength G, normalized as in a previous footnote). The supergravity Lagrangian, up to terms quartic in the gravitino (which we will not need), is [8]

$$\begin{aligned}L_S=\frac{1}{\kappa^2}\int_{M^{11}}d^{11}x\sqrt{g}\Big(&-\frac{1}{2}R-\frac{1}{2}\overline{\psi}_I\Gamma^{IJK}D_J\psi_K-\frac{1}{48}G_{IJKL}G^{IJKL}\\&-\frac{\sqrt{2}}{192}\left(\overline{\psi}_I\Gamma^{IJKLMN}\psi_N+12\overline{\psi}^J\Gamma^{KL}\psi^M\right)G_{JKLM}\\&-\frac{\sqrt{2}}{3456}\epsilon^{I_1I_2\ldots I_{11}}C_{I_1I_2I_3}G_{I_4\ldots I_7}G_{I_8\ldots I_{11}}\Big).\end{aligned}\tag{2.1}$$

We work in 1.5 order formalism: the spin connection Ω is formally regarded as an independent variable, and eventually set equal to the solution of the bulk equations of motion. The Riemann tensor is the field strength constructed from Ω.

The transformation laws of local supersymmetry read

$$\begin{aligned} \delta e_I{}^m &= \frac{1}{2}\overline{\eta}\Gamma^m\psi_I \\ \delta C_{IJK} &= -\frac{\sqrt{2}}{8}\overline{\eta}\Gamma_{[IJ}\psi_{K]} \\ \delta\psi_I &= D_I\eta + \frac{\sqrt{2}}{288}\left(\Gamma_I{}^{JKLM} - 8\delta_I^J\Gamma^{KLM}\right)\eta G_{JKLM} + \dots . \end{aligned} \tag{2.2}$$

(The ... are three fermi terms in the transformation law of ψ, often absorbed in a definition of "supercovariant" objects; we will not need them.)

We suppose that there is a $\mathbf{Z}_2$ symmetry acting on M^{11}, with codimension one fixed points. We let M^{10} be a component of the fixed point set; we will study the physics near M^{10}. We suppose that the fields are required to be invariant under the $\mathbf{Z}_2$; this means that we could pass to the manifold-with-boundary $X = M^{11}/\mathbf{Z}_2$ (with boundary M^{10}), but that will not be particularly convenient. If M^{10} is defined locally by an equation $x^{11} = 0$, x^{11} being one of the coordinates (and the $\mathbf{Z}_2$ acting by $x^{11} \to -x^{11}$), then (with an appropriate lifting of the $\mathbf{Z}_2$ action to spinors and the three-form) the supersymmetries that commute with the $\mathbf{Z}_2$ action are generated by spinor fields η on M^{11} that obey

$$\Gamma_{11}\eta = \eta \ \text{ at } x^{11} = 0. \tag{2.3}$$

$\mathbf{Z}_2$ invariance of the gravitino means that

$$\begin{aligned} \Gamma_{11}\psi_A &= \psi_A, \qquad A = 1, \dots, 10 \\ \Gamma_{11}\psi_{11} &= -\psi_{11}. \end{aligned} \tag{2.4}$$

As in (2.4), we will use $A, B, C, D = 1, \dots, 10$ for indices tangent to M^{10}. For the three-form C, because it is odd under parity (this follows from the CGG interaction in (2.1)), $\mathbf{Z}_2$ invariance means that $C_{BCD} = 0$ at $x^{11} = 0$. A gauge-invariant statement that follows from this is that

$$G_{ABCD} = 0 \ \text{ at } x^{11} = 0, \tag{2.5}$$

or in other words, the pull-back of the differential form G to M^{10} vanishes. We will eventually find a sort of modification of this statement in order $\kappa^{2/3}$.

The vector supermultiplets, which propagate on M^{10}, consist of the E_8 gauge field A (with field strength $F_{CD} = \partial_C A_D - \partial_D A_C + [A_C, A_D]$) and fermions (gluinos) χ in the adjoint representation, obeying $\Gamma_{11}\chi = \chi$. The minimal Yang-Mills Lagrangian is

$$L_{YM} = -\frac{1}{\lambda^2}\int_{M^{10}} d^{10}x\sqrt{g}\ \mathrm{tr}\left(\frac{1}{4}F_{AB}F^{AB} + \frac{1}{2}\overline{\chi}\Gamma^A D_A\chi\right). \tag{2.6}$$

(Here $d^{10}x\sqrt{g}$ is understood as the Riemannian measure of M^{10}, using the restriction to M^{10} of the metric on M^{11}.) In (2.6), λ is the gauge coupling constant. We will ultimately see that $\lambda \sim \kappa^{2/3}$, so that L_{YM} is of order $\kappa^{2/3}$ relative to L_S. The supersymmetry transformation laws are

$$\begin{aligned}\delta A_A^a &= \frac{1}{2}\overline{\eta}\Gamma_A\chi^a \\ \delta\chi^a &= -\frac{1}{4}\Gamma^{AB}F_{AB}^a\eta.\end{aligned} \tag{2.7}$$

We have here made explicit an index $a = 1, \dots, 248$ labeling the adjoint representation of E_8. We define an inner product by $X^aX^a = \mathrm{tr}\, X^2 = (1/30)\mathrm{Tr}X^2$, with Tr the trace in the adjoint representation.

We wish to add additional interactions to the above and modify the supersymmetry transformation laws so that $L_S + L_{YM} + \dots$ will be locally supersymmetric. The first steps are as follows. Let T_{YM} and $\mathcal{S}_{YM}$ be the energy-momentum tensor and supercurrent of the supergauge multiplet. As in any coupling of matter to supergravity, the variation of L_{YM} under local supersymmetry contains terms $D_A\overline{\eta}\mathcal{S}_{YM}^A$, reflecting the fact that L_{YM} is only invariant under (2.7) if η is covariantly constant, and $\overline{\eta}\psi T_{YM}$, coming from the variation of L_{YM} under a local supersymmetry transformation of the metric. To cancel these variations, it is necessary – as usual in supergravity – to add an interaction $\overline{\psi}\mathcal{S}_{YM}$. In the case at hand, this interaction is

$$L_1 = -\frac{1}{4\lambda^2}\int_{M^{10}} d^{10}x\sqrt{g}\ \overline{\psi}_A\Gamma^{BC}\Gamma^A F_{BC}^a\chi^a. \tag{2.8}$$

A small calculation shows that the variation of L_1 under local supersymmetry cancels the $D\overline{\eta}\chi F$ and $\eta\psi\chi D\chi$ variations of L_{YM}, and also cancels part of the $\eta\psi F^2$ variation. The uncanceled variation turns out, after some gamma matrix gymnastics, to be

$$\Delta = \frac{1}{16\lambda^2}\int_{M^{10}} d^{10}x\sqrt{g}\ \overline{\psi}_A\Gamma^{ABCDE}F_{BC}^aF_{DE}^a\eta. \tag{2.9}$$

Rather as in the coupling of the ten-dimensional vector multiplet to *ten*-dimensional supergravity [2], there is no way to cancel this variation by adding to the Lagrangian

additional matter couplings. A peculiar mixing of the supergravity and matter multiplets is needed.

When one verifies the local supersymmetry of the eleven-dimensional supergravity Lagrangian L_S, it is necessary among other things to check the cancellation of the $\eta GD\psi$ and $D\eta G\psi$ terms. In this verification, it is necessary to integrate by parts and use the Bianchi identity $dG = 0$.[2] To cancel Δ, one must modify the Bianchi identity to read

$$dG_{11\,ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})F^a_{[AB}F^a_{CD]}. \tag{2.10}$$

This correction to the Bianchi identity adds an extra variation of L_S that precisely cancels Δ.

Much as in the analogous story in ten dimensions, (2.10) implies that the three-form C is not invariant under Yang-Mills gauge transformations. To determine the gauge transformation law of C, it is convenient to solve the modified Bianchi identity by introducing

$$\omega_{BCD} = \mathrm{tr}\left(A_B(\partial_C A_D - \partial_D A_C) + \frac{2}{3}A_B[A_C, A_D] + \text{cyclic permutations of } B, C, D\right). \tag{2.11}$$

Thus

$$\partial_C\omega_{BCD} + \text{cyclic permutations} = 6\ \mathrm{tr}\, F_{[AB}F_{CD]}. \tag{2.12}$$

The Bianchi identity can then be solved by modifying the definition of $G_{11\,ABC}$, the new definition being

$$G_{11\,ABC} = (\partial_{11}C_{ABC} \pm 23 \text{ permutations}) + \frac{\kappa^2}{\sqrt{2}\lambda^2}\delta(x^{11})\omega_{ABC}. \tag{2.13}$$

Under an infinitesimal gauge transformation $\delta A^a_A = -D_A\epsilon^a$, ω transforms by

$$\delta\omega_{ABC} = \partial_A\left(\mathrm{tr}\ \epsilon F_{BC}\right) + \text{cyclic permutations of } A, B, C, \tag{2.14}$$

so gauge invariance of $G_{11\,ABC}$ holds precisely if the three-form C transforms under gauge transformations by

$$\delta C_{11\,AB} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2}\delta(x^{11})\ \mathrm{tr}\ \epsilon F_{AB}. \tag{2.15}$$

[2] This occurs when one varies the interaction $\overline{\psi}_I\Gamma^{IJKLMN}\psi_N G_{JKLM}$ with $\delta\overline{\psi}_I \sim D_I\overline{\eta}$. To cancel other variations, one must integrate by parts so that the D_I acts on ψ_N instead of on η. The integration by parts gives a term proportional to dG_{IJKLM}.

A correction to the supersymmetry transformation law of $C_{11\,BC}$ is also necessary. It can be determined by requiring that the supersymmetry variation of $G_{11\,ABC}$ be gauge-invariant (otherwise this variation gives gauge non-invariant, uncancellable terms in the supersymmetry variation of the Lagrangian) and is

$$\widetilde{\delta}C_{11\,BC} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2}\mathrm{tr}\,(A_B\delta A_C - A_C\delta A_B)\,, \tag{2.16}$$

where on the right δA is the standard supergravity transformation law given in (2.7). With this correction to δC, the correction to δG is

$$\widetilde{\delta}G_{11\,ABC} = \frac{\kappa^2}{\sqrt{2}\lambda^2}\delta(x^{11})\,(\overline{\eta}\Gamma_A\chi^a F^a_{BC} + \text{cyclic permutations of } A, B, C)\,. \tag{2.17}$$

Boundary Behavior

There is another sense in which we can "solve the Bianchi identity." We can ask, compatibly with the equation of motion, what can be the behavior near $x^{11} = 0$ of a G field that obeys the corrected Bianchi identity found above, which was

$$dG_{11\,ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})F^a_{[AB}F^a_{CD]}. \tag{2.18}$$

How can dG acquire such a delta function? G itself cannot have a delta function at $x^{11} = 0$, as that would not be compatible with the equations of motion. However, as G is odd under $x^{11} \to -x^{11}$, it is natural for G_{ABCD} to have a step function discontinuity at $x^{11} = 0$, giving a delta function in dG. In fact, G_{ABCD} must have a jump at $x^{11} = 0$ given precisely by

$$G_{ABCD} = -\frac{3}{\sqrt{2}}\frac{\kappa^2}{\lambda^2}\epsilon(x^{11})F^a_{[AB}F^a_{CD]} + \dots. \tag{2.19}$$

Here $\epsilon(x^{11})$ is 1 for $x^{11} > 0$ and -1 for $x^{11} < 0$; the $\dots$ are terms that are regular near $x^{11} = 0$ and therefore (since G is odd under $x^{11} \to -x^{11}$) vanish at $x^{11} = 0$. This is the behavior required by the modified equations of motion and Bianchi identity.

This discontinuity means that G_{ABCD} does not have a well-defined limiting value as $x^{11} \to 0$. However, G^2 has such a limit, which moreover is determined by (2.19) in terms of the gauge fields at $x^{11} = 0$.

There is another interesting way to think about (2.19). In this paper we are working "upstairs" on a smooth eleven-manifold M^{11}, and requiring $\mathbf{Z}_2$ invariance. It is natural conceptually (though sometimes less convenient computationally) to work "downstairs"

on the manifold-with-boundary $X = M^{11}/\mathbf{Z}_2$. In that case, it is not natural to add a correction to dG supported at the boundary of X (that is, at $x^{11} = 0$). More natural is to impose a boundary condition that has the same effect. Assuming that one identifies X with the portion of M^{11} with $x^{11} > 0$, the requisite boundary condition is simply

$$G_{ABCD}|_{x^{11}=0} = -\frac{3}{\sqrt{2}}\frac{\kappa^2}{\lambda^2}F^a_{[AB}F^a_{CD]}. \tag{2.20}$$

(If one identifies X with the $x^{11} < 0$ portion of M^{11}, one would want the opposite sign in (2.20).) The idea here is that, since $dG = 0$, the integration by parts explained in the footnote just before (2.10) no longer picks up a delta function term, but (since there is now a boundary) it does pick up a boundary term that has the same effect.[3]

Thus, in working downstairs on X, G has a well-defined boundary value given by (2.20) (or the same expression with opposite sign if one picks orientations opppositely). In working upstairs on M, G does not quite have a well-defined value at $x^{11} = 0$, but G^2 does.

3. Analysis of Anomalies

The most important conclusions of the last section are the gauge transformation law (2.15) for the three-form C, and the formula (2.19) for the behavior of G near M^{10}. We will now put these together to get an eleven-dimensional view of gauge and gravitational anomalies.

The idea is that (2.15) is analogous to the gauge transformation law $\delta B \sim \mathrm{tr}\,\epsilon F$ for the two-form B of string theory, and (2.19) will turn the "Chern-Simons interaction" $\int C \wedge G \wedge G$ of eleven-dimensional supergravity into a Green-Schwarz term.

We recall that the CGG interaction is, to be precise, a term

$$W = -\frac{\sqrt{2}}{3456\kappa^2}\int_{M^{11}} \epsilon^{M_1M_2\ldots M_{11}}C_{M_1M_2M_3}G_{M_4\ldots M_7}G_{M_8\ldots M_{11}}. \tag{3.1}$$

The variation of W under an arbitrary variation of C is therefore

$$\delta W = -\frac{\sqrt{2}}{1152\kappa^2}\int_{M^{11}} \epsilon^{M_1M_2\ldots M_{11}}\delta C_{M_1M_2M_3}G_{M_4\ldots M_7}G_{M_8\ldots M_{11}}. \tag{3.2}$$

[3] In working on $X = M^{11}/\mathbf{Z}_2$ instead of M^{11}, one should replace the $1/\kappa^2$ in (2.1) by $2/\kappa^2$, because one is integrating (2.1) over a space of half the volume. This factor of 2 goes into verifying the normalization of (2.20).

Given that C is not invariant under gauge transformations, neither is W. Using (2.15) for the gauge variation of C, we get for the gauge variation of W

$$\delta W = -\frac{1}{2304\lambda^2}\int_{M^{10}} \epsilon^{M_1M_2\dots M_{10}}\epsilon^a F^a_{M_1M_2}G_{M_3\dots M_6}G_{M_7\dots M_{10}}. \tag{3.3}$$

To proceed further, we need the value of G^2 at $x^{11}=0$. This is given by (2.19) in the orbifold approach or equivalently by the boundary condition (2.20) if one works "downstairs." Either way, one gets

$$\delta W = -\frac{\kappa^4}{128\lambda^6}\int_{M^{10}} \epsilon^{M_1M_2\dots M_{10}}\epsilon^a F^a_{M_1M_2}F^b_{M_3M_4}F^b_{M_5M_6}F^c_{M_7M_8}F^c_{M_9M_{10}}. \tag{3.4}$$

So, as promised, the classical theory is not gauge invariant. There is no way to cure this at the classical level. The only recourse is to quantum anomalies. The anomalous variation of the effective action Γ for ten-dimensional Majorana-Weyl fermions in an arbitrary representation of a simple gauge group is

$$\delta\Gamma = \frac{1}{2}\frac{1}{(4\pi)^5 5!}\int_{M^{10}} \epsilon^{M_1M_2\dots M_{10}}\mathrm{Tr}\left(\epsilon F_{M_1M_2}F_{M_3M_4}\dots F_{M_9M_{10}}\right), \tag{3.5}$$

with Tr being the trace in the fermion representation. The case that we are interested in is that the gauge group is E_8 and the fermions are in the adjoint representation. In that case, one has the wonderful and unique (to E_8) identity $\mathrm{Tr}W^6 = (\mathrm{Tr}W^2)^3/7200$ (and likewise $\mathrm{Tr}\epsilon F^5 = \mathrm{Tr}\epsilon F(\mathrm{Tr}F^2)^2/7200$). If furthermore we write, as is customary, $\mathrm{tr}\,W^2 = \mathrm{Tr}W^2/30$, then $\mathrm{Tr}W^6 = (15/4)(\mathrm{tr}\,W^2)^3$. In this case, therefore, the quantum anomaly (3.5) can be written

$$\delta\Gamma = \frac{15}{8(4\pi)^5 5!}\int_{M^{10}} \epsilon^{M_1M_2\dots M_{10}}\,\mathrm{tr}\left(\epsilon F_{M_1M_2}\right)\mathrm{tr}\left(F_{M_3M_4}F_{M_5M_6}\right)\mathrm{tr}\left(F_{M_7M_8}F_{M_9M_{10}}\right). \tag{3.6}$$

It therefore has the right structure to cancel (3.4) (recall that the metric on the Lie algebra was defined by $\epsilon^a F^a = \mathrm{tr}\,\epsilon F$).

Implementing this cancellation, we learn finally that, as promised in the introduction, the gauge coupling is related to the gravitational coupling by

$$\lambda^2 = 2\pi\left(4\pi\kappa^2\right)^{2/3}. \tag{3.7}$$

One might have expected that the analogs of the Green-Schwarz terms in the present discussion would be boundary interactions, that is interactions supported at $x^{11}=0$. This is not the case, as we have seen. In fact, given a gauge variation of C proportional to $\delta(x^{11})$, the possible resulting gauge variation of a boundary interaction would necessarily be proportional to $\delta(0)$. Thus, the "Green-Schwarz terms" must be bulk interactions; this goes for the "Chern-Simons" CGG term and other terms discussed below.

3.1. Extension to Gravitational Anomalies

We determined the gauge coupling by canceling the purely gauge anomalies at the boundary of the eleven-dimensional world. We would now like to include also the gravitational and mixed anomalies.

From the above discussion, the anomaly four-form $\widehat{I}_4$ is the four-form that appears (multiplied by $\delta(x^{11})dx^{11}$) in the Bianchi identity for G. In our work so far, we have seen only a $\mathrm{tr}\, F^2$ term in $\widehat{I}_4$, but in view of the known form of the ten-dimensional anomalies, the actual structure must be $\mathrm{tr}\, F^2 - (1/2)\mathrm{tr}\, R^2$. Thus, the modified Bianchi identity (2.10) should be replaced by

$$dG_{11\,ABCD} = -3\sqrt{2}\frac{\kappa^2}{\lambda^2}\delta(x^{11})\left(F^a_{[AB}F^a_{CD]} - \frac{1}{2}\mathrm{tr}\, R_{[AB}R_{CD]}\right), \tag{3.8}$$

and the formula (2.19) for the behavior near $x^{11} = 0$ should correspondingly be replaced by

$$G_{ABCD} = -\frac{3}{\sqrt{2}}\frac{\kappa^2}{\lambda^2}\epsilon(x^{11})\left(F^a_{[AB}F^a_{CD]} - \frac{1}{2}\mathrm{tr}\, R_{[AB}R_{CD]} + \dots\right). \tag{3.9}$$

There is also a corresponding local Lorentz transformation law of (3.9), analogous to the E_8 gauge transformation law (2.15).

The $\mathrm{tr}\, R^2$ terms in these formulas are not required by the low energy supergravity, but (since they are needed for anomaly cancellation, given the structure of the one-loop chiral anomalies), they must be present in the full M-theory. The situation is presumably analogous to what is seen for the perturbative heterotic string, where the $\mathrm{tr}\, R^2$ terms in the analogous formulas arise as corrections of order α'. Note that the $\mathrm{tr}\, R^2$ correction will appear in (3.8) and (3.9) with the same coefficient, since (3.9) is deduced from (3.8) by reasoning that was explained above.

Having understood how $\widehat{I}_4$ enters in eleven dimensions, we would like now to understand the origin of $\widehat{I}_8$, or equivalently to complete our understanding of the Green-Schwarz terms. We have already found one of the Green-Schwarz terms above – the long-familiar "Chern-Simons" interaction of eleven-dimensional supergravity. This particular interaction gives a contribution to $\widehat{I}_8$ that is a multiple of $\widehat{I}_4^2$, since the boundary behavior of G is $G \sim \widehat{I}_4$, as we have seen. The other Green-Schwarz terms will have to be bulk interactions, as explained at the end of the last subsection, and more precisely will have to be interactions of the form

$$\mathcal{I} = \int_{M^{11}} C \wedge \left(a\,\mathrm{tr}\, R^4 + b(\mathrm{tr}\, R^2)^2\right), \tag{3.10}$$

these being the terms that have the right sort of gauge and local Lorentz variations to cancel chiral anomalies. Note that it is impossible to add to (3.10) terms that directly involve F, since the gauge fields propagate only on M^{10}. It is also impossible for F-dependence to arise indirectly from the behavior of G near M^{10}, since (3.10) is independent of G; the $C\wedge G\wedge G$ has already been taken into account (with a coefficient known from low energy supergravity), and a term $C\wedge G\wedge \mathrm{tr}\, R^2$ is not possible, as it would violate the parity symmetry of M-theory.

$\mathcal{I}$ will contribute to $\widehat{I}_8$ a term that involves R only, so we get the striking prediction that $\widehat{I}_8$ is a multiple of $\widehat{I}_4^2$ plus an eight-form constructed only from R. At this point, it is helpful to note that

$$\widehat{I}_8=-\frac{1}{4}\left(\mathrm{tr}\, F^2-\frac{1}{2}\mathrm{tr}\, R^2\right)^2+\left(-\frac{1}{8}\mathrm{tr}\, R^4+\frac{1}{32}(\mathrm{tr}\, R^2)^2\right), \tag{3.11}$$

and thus has the expected form. This is a satisfying test of M-theory, as this structure of $\widehat{I}_8$ has no known rationale in perturbative string theory.

Actually, we can be more precise, since the interaction (3.10) is known (at least up to an overall multiplicative constant; fixing this constant requires a more precise comparison of the normalizations of string theory and M-theory or a precise knowledge of the two-brane and five-brane tensions in M-theory). As we explained in the introduction, the interaction (3.10) is known up to a constant multiple either from comparison to a one-loop calculation for Type IIA superstrings [7] or from anomaly cancellation for eleven-dimensional five-branes [4-6]. Either way, one finds that (3.10) is a multiple of

$$\frac{\sqrt{2}}{(4\pi)^3(4\pi\kappa^2)^{1/3}}\int_{M^{11}}C\wedge\left(-\frac{1}{8}\mathrm{tr}\, R^4+\frac{1}{32}(\mathrm{tr}\, R^2)^2\right). \tag{3.12}$$

Thus, at least the relative coefficient of $\mathrm{tr}\, R^4$ and $(\mathrm{tr}\, R^2)^2$ agrees with the "experimental" structure of $\widehat{I}_8$. This is again a real test of M-theory, since there is no perturbative string theory reason for this to work. The structure of (3.10) is deduced either via Type IIA perturbation theory or anomaly cancellation for eleven-dimensional five-branes, and any known rationale for comparing the anomaly polynomial of the perturbative heterotic string to either of these involves M-theory.

Notice that the coefficient $-1/4$ of the $\widehat{I}_4^2$ term in $\widehat{I}_8$ is a matter of convention; it could be shifted by a scaling $\widehat{I}_4\to u\widehat{I}_4$, $\widehat{I}_8\to u^{-1}\widehat{I}_8$, without affecting the factorization $\widehat{I}_{12}=\widehat{I}_4\widehat{I}_8$. Modulo this imprecision in the definition of $\widehat{I}_8$, we have from M-theory a

complete *a priori* prediction for $\widehat{I}_8$, which amounts to a prediction for three numbers ($\widehat{I}_8$ is a linear combination of four monomials $(\mathrm{tr}\, F^2)^2$, $\mathrm{tr}\, F^2\, \mathrm{tr}\, R^2$, $(\mathrm{tr}\, R^2)^2$, and $\mathrm{tr}\, R^4$, but the coefficient of one monomial can be scaled out as just explained). There are therefore three predictions, of which we have here verified two; verification of the last prediction requires a more precise comparison of different conventions, as noted in the last paragraph.

4. Construction of the Lagrangian

In this section, we will proceed with additional steps in the construction of the locally supersymmetric Lagrangian. The formula $\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}$ obtained in the last section is of some conceptual interest in organizing the computation. It shows that the theory has only one natural length scale, given by $\kappa^{2/9}$. Moreover, on dimensional grounds, the decomposition of the boundary interactions in terms with more and more matter fields is an expansion in powers of κ. The leading boundary interactions (the minimal Lagrangian L_{YM} of the gauge multiplet and terms related to it by supersymmetry) are of order $\kappa^{2/3}$ relative to the gravitational action. Formally, the construction of the locally supersymmetric classical action appears to be an expansion in integral powers of $\kappa^{2/3}$. Other exponents must arise in the actual quantum M-theory, since we will run into infinities which, when cut off in the quantum theory, must on dimensional grounds give anomalous powers of κ.

There are two principal goals of the rather complicated computation performed in this section:

(1) To add to our confidence that the supersymmetric coupling of the vector multiplet on M^{10} to the supergravity multiplet on M^{11} does exist, by working out the classical construction of this coupling to the extent that it makes sense.

(2) To exhibit the limits of the classical construction (beyond what is evident from the discussion of anomalies in section three) by showing how infinities arise in order $\kappa^{4/3}$.

In the computation, one can be guided to a certain extent by the *ten*-dimensional coupling of the vector and supergravity multiplets [2,3], to which our discussion must reduce at low energies in the appropriate limit. This gives clues to many of the terms that must be added to the Lagrangian and transformation laws. On the other hand, in the computation one definitely meets terms (involving $D_{11}\eta$, for instance) that vanish upon dimensional reduction to ten dimensions but must be canceled to achieve local supersymmetry in eleven

dimensions. Thus, the existence of the coupling we are constructed (and again, we believe that it only exists in full at the quantum level) goes well beyond ten-dimensional considerations.

We will carry out the computation in three stages: (i) first we complete the construction of the Lagrangian in order $\kappa^{2/3}$; (ii) then we look at some terms in order $\kappa^{4/3}$; (iii) finally we look systematically at all four-fermi variations in order $\kappa^{2/3}$. It might seem illogical to put (ii) before (iii). We have done this because (ii) is much simpler than (iii), and also gives an easy way to determine some of the transformation laws that are needed in (iii).

Of course, we cannot hope for a full determination of the structure. Apart from requiring a much fuller knowledge of the quantum mechanics of M-theory than one has, the full structure is presumably non-polynomial, like the α' expansion of perturbative string theory. Once one reaches a sufficiently high order in κ, one would require among other things a more complete knowledge of the low energy expansion of M-theory in bulk (including higher derivative interactions) in order to proceed.

4.1. Some New Interactions

The boundary interactions (that is, interactions supported at $x^{11} = 0$) that we discussed in section two are the minimal super Yang-Mills action and the supercurrent coupling

$$L_0 = -\frac{1}{2\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}x\sqrt{g}\,\left(\frac{1}{4}\,\mathrm{tr}\,F_{AB}F^{AB} + \frac{1}{2}\,\mathrm{tr}\,\overline{\chi}\Gamma^A D_A\chi + \frac{1}{4}\overline{\psi}_A\Gamma^{BC}\Gamma^A F^a_{BC}\chi^a\right). \tag{4.1}$$

These terms are all of order $\kappa^{2/3}$ compared to the supergravity action. There is precisely one more boundary interaction of the same order. To find it, one can look at the terms of order $F\chi G\eta$ in the supersymmetry variation of the Lagrangian. One source of such terms comes from the variation of the supercurrent interaction in (4.1) with $\delta\psi \sim G\eta$. Another source comes as follows. We found in section two a correction (2.17) to the supersymmetry variation of $G_{ABC\,11}$. The $G^2_{ABC\,11}$ term in the bulk supergravity action therefore picks up a new variation supported at $x^{11} = 0$; this term is again proportional to $F\chi G\eta$. These terms by themselves do not cancel. After a moderately lengthy computation, one finds that to cancel them one must add a new boundary interaction,

$$L_1 = \frac{\sqrt{2}}{96\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}x\sqrt{g}\;\overline{\chi}^a\Gamma^{ABC}\chi^a G_{ABC\,11}. \tag{4.2}$$

This term – and the verification that the $F\chi G\eta$ terms cancel – is quite similar to an analogous term and verification in the ten-dimensional supergravity/Yang-Mills coupling.

This actually completes the construction of the Lagrangian in order $\kappa^{2/3}$ and verification of local supersymmetry up to four-fermi terms, whose analysis we postpone to the next subsection. Instead we turn to something that is of conceptual interest and still relatively simple.

In (4.2), we see an interaction in which $G_{ABC\,11}$ is evaluated on M^{10}, that is at $x^{11} = 0$. On the other hand, in (2.17), we found a term in the supersymmetry variation of $G_{ABC\,11}$ that is proportional to $\delta(x^{11})$. If we combine the two, that is if we vary L_1 according to (2.17), we get a result proportional to $\delta(0)$. This presumably should be interpreted as a linear divergence that is cut off somehow in the quantum M-theory. For our present purposes, though, we will be pragmatic, and without worrying about precisely what $\delta(0)$ means, we will attempt to formally cancel the $\delta(0)$ terms.

Obviously, to do this we need more sources of $\delta(0)$ terms. Since the term we want to cancel is proportional to $\chi\chi\chi F\eta$, there are two sources of terms that might cancel it. We could add to the gravitino variation an extra term $\delta\psi_A \sim \delta(x^{11})\chi\chi\eta$. When combined with the $\chi F\psi$ interaction in L_0, it gives another term of the general form $\delta(0)\chi^3 F\eta$. Finally, one could add to the Lagrangian a term $\delta(0)\chi^4$, which will again have a variation of the desired form. After another moderately long calculation, one finds that the new terms required in the gravitino variation are[4]

$$\delta\psi_A = -\frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta(x^{11})\,\overline{\chi}^a\Gamma_{BCD}\chi^a\left(\Gamma_A{}^{BCD}\eta - 6\delta_A^B\Gamma^{CD}\eta\right), \tag{4.3}$$

and that the new interaction required is

$$L_\chi = -\frac{\delta(0)}{96(4\pi)^{10/3}\kappa^{2/3}}\int_{M^{10}} d^{10}x\sqrt{g}\;\overline{\chi}^a\Gamma^{ABC}\chi^a\,\overline{\chi}^b\Gamma_{ABC}\chi^b. \tag{4.4}$$

The $\delta(0)$ presumably means that in the quantum theory this interaction is really of order $\kappa^{-8/9}$, that is, of order $\kappa^{10/9}$ relative to the original supergravity action.

We focussed here on a particular four-fermi term of relative order $\kappa^{4/3}$ because it enabled us to exhibit in a simple fashion the "divergences" that appear in trying to construct the classical Lagrangian. In the next section, we look systematically at the four-fermi terms of order $\kappa^{2/3}$.

[4] Here $\delta(x^{11})$ is understood as the delta function that transforms as a scalar under diffeomorphisms; this involves an implicit power of the 11-11 component of the frame field $e_{11}{}^{11}$, and is required to match the transformation properties of $\delta\psi_A$.

4.2. Four-Fermi Terms in Order $\kappa^{2/3}$

To this order, the structure of the Lagrangian and the supersymmetry variation of the fields can be determined by canceling terms $\sim \chi\chi\psi\eta$ with one covariant derivative acting on one of the fermi fields. There are two natural classes of such terms, depending on whether the derivative is normal to the boundary, or acts along the boundary.

First consider the class of terms containing the normal derivative, i.e. terms proportional to $D_{11}\eta$ or $D_{11}\psi_A$. Such terms clearly vanish upon the dimensional reduction to ten dimensions: from the point of view of a ten-dimensional low-energy observer, they only contain contributions from massive Kaluza-Klein modes of ψ_A and η that decouple from the low-energy modes as the radius of the eleventh dimension goes to zero. In the present case, however, the cancellation of such terms is not automatic, and will help us determine some new additions to the Lagrangian.

What are the possible sources of terms proportional to $D_{11}\psi_A$ or $D_{11}\eta$? One source of such terms is generated by the correction (4.3) to the variation of ψ_A determined in the previous subsection. Since this correction is proportional to the delta function localized at the boundary, it will generate boundary four-fermi terms proportional to $D_{11}\psi_A$ when applied to the bulk kinetic term of the gravitino. Given (4.3), this variation generates just one term, equal to

$$-\frac{1}{64\pi(4\pi\kappa^2)^{2/3}}\int_{M^{10}} d^{10}x\sqrt{g}\;\overline{\chi}^a\Gamma^{ABC}\chi^a\;\overline{\eta}\Gamma_{AB}D_{11}\psi_C. \tag{4.5}$$

This term cancels exactly against a similar term that comes from the bulk variation of $G_{ABC\,11}$ in the interaction term L_1. No other terms with $D_{11}\psi_A$ appear in the supersymmetry variation of the Lagrangian at this order.

As to the terms with $D_{11}\eta$, they have two sources among the terms already present in the Lagrangian. First of all, the bulk variation of $G_{ABC\,11}$ in L_1 produces a term proportional to $\overline{\chi}\Gamma^{ABC}\chi\;\overline{\psi}_A\Gamma_{BC}D_{11}\eta$. This term can only be canceled if we introduce a new interaction,

$$L_2 = \frac{1}{64\pi(4\pi\kappa^2)^{2/3}}\int_{M^{10}} d^{10}x\sqrt{g}\;\overline{\chi}^a\Gamma^{ABC}\chi^a\;\overline{\psi}_A\Gamma_{BC}\psi_{11}. \tag{4.6}$$

(When L_2 is varied, $\delta\psi_A \sim D_A\eta$ leads to a $\chi\chi\psi_{11}D_A\eta$ term; this term cancels against the term of the same form that comes from the variation of $G_{ABC\,11}$ in L_1. In addition, the variations of the bulk supergravity action L_S and of L_1 give terms of the form $\chi\chi\eta D_A\psi_{11}$; happily, these terms cancel against each other.)

Another source of terms proportional to $D_{11}\eta$ is the variation of the spin connection in the gaugino kinetic term. This point requires a further explanation. In our treatment of eleven-dimensional supergravity in bulk, we have adopted the 1.5 order formalism, which means that the spin connection is first treated as an independent variable and set equal to the solution of its bulk equations of motion at the end of the calculation. One then need not worry about the supersymmetry variation of the spin connection, which vanishes by the equations of motion. In including the boundary interactions, we prefer to continue to use the "bulk" formula for the spin connection. This means that when the spin connection appears in boundary interactions, its supersymmetry variation must be included. (One could avoid this by extending the 1.5 order formalism to incorporate boundary corrections to the spin connection determined by the equations of motion, but we did not find that approach simpler.)

In practice, to the order we will calculate, the spin connection only appears in the gravitino kinetic term. Its variation produces an additional term proportional to $D_{11}\eta$ (plus other terms we will consider later). Canceling this $D_{11}\eta$ term requires a new term in the Lagrangian,

$$L_3 = \frac{1}{64\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}x\sqrt{g}\ \overline{\chi}^a \Gamma_{ABC}\chi^a\ \overline{\psi}_D \Gamma^{DABC}\psi_{11}. \tag{4.7}$$

Now we will show that no other ψ_{11} dependent four-fermi terms are generated in the Lagrangian at this order in κ, beyond those given by (4.6) and (4.7). To see this, we will proceed as follows. Supersymmetry variation of such additional four-fermi terms would produce additional terms proportional to $D_{11}\eta$. Notice that these $D_{11}\eta$ terms could only be canceled if there is a three-fermi correction to the variation of the gravitino, $\delta'\chi \sim \chi\psi_{11}\eta$, and the gaugino kinetic term is varied. We will prove our claim that no new ψ_{11}-dependent four-fermi terms arise in the Lagrangian at this order, by showing that there are no $\chi\psi_{11}\eta$ corrections to the supersymmetry variation of χ. Upon variation of the gaugino kinetic term, such corrections would produce terms of the form

$$\overline{\chi}\Gamma_A D_B\chi\,\overline{\psi}_{11}\ldots\eta \quad \text{and} \quad \overline{\chi}\Gamma_{ABCDE}D_F\chi\,\overline{\psi}_{11}\ldots\eta. \tag{4.8}$$

(Here $\ldots$ denotes all possible combinations of Γ matrices.) There is no other possible source of such terms; a simple calculation shows that their cancellation requires the supersymmetry variation of χ^a to be independent of ψ_{11}, thus completing our argument.

Having canceled all terms with $D_{11}\psi$ and $D_{11}\eta$, we can determine the rest of the structure at this order in κ by looking at cancellations of $\chi\chi\eta\psi$ terms where now the *ten*-dimensional derivative D_A acts on one of the four fermions. First we determine the correction to the supersymmetry variation of χ^a, by canceling terms of the form

$$\overline{\chi}\Gamma_A D_B\chi\,\overline{\eta}\ldots\psi_C \quad \text{and} \quad \overline{\chi}\Gamma_{ABCDE}D_F\chi\,\overline{\eta}\ldots\psi_G. \tag{4.9}$$

Terms of this structure must cancel by themselves, since chirality and fermi statistics do not allow one to use integration by parts to move the derivative away from the gauginos. There are two obvious sources of such terms: the variation of $e_A{}^m$ in the gaugino kinetic term, and the variation of F^a_{AB} in the supercurrent coupling of L_0. As these do not cancel, one has to look for another source of such terms. We can add a correction, $\delta'\chi \sim \chi\psi\eta$, to the supersymmetry variation of the gauginos. This correction will produce terms of the required form (4.9) from the variation of the gaugino kinetic term, and the precise form of $\delta'\chi$ will be determined from the cancellation of these terms.[5] After a tedious calculation, one obtains

$$\begin{aligned}\delta'\chi^a = \frac{1}{64}\Big(&7\left(\overline{\psi}_A\Gamma_B\eta\right)\Gamma^{AB}\chi^a + 9\left(\overline{\psi}_A\Gamma^A\eta\right)\chi^a - \frac{1}{2}\left(\overline{\psi}_A\Gamma_{BCD}\eta\right)\Gamma^{ABCD}\chi^a \\ &- \frac{5}{2}\left(\overline{\psi}^A\Gamma_{ABC}\eta\right)\Gamma^{BC}\chi^a + \frac{1}{24}\left(\overline{\psi}^A\Gamma_{ABCDE}\eta\right)\Gamma^{BCDE}\chi^a\Big).\end{aligned} \tag{4.10}$$

The correction (4.10) to the supersymmetry variation of χ^a can be simplified considerably by the Fierz rearrangement formula, leading to

$$\delta'\chi^a = \frac{1}{4}\left(\overline{\psi}_A\Gamma_B\chi^a\right)\Gamma^{AB}\eta. \tag{4.11}$$

This is exactly what one would have expected from the requirement that the total supersymmetry transformation of χ^a be "supercovariant." This also explains why no ψ_{11}-dependent corrections to the supersymmetry variation of χ^a arise – when varied, such

[5] In this and some of the following calculations, we need a Fierz rearrangement formula for chiral ten-dimensional fermions. All rules follow from the expansion of the product of two fermions ξ and ζ on M^{10} that obey $\Gamma_{11}\xi = \xi$ and $\Gamma_{11}\zeta = \zeta$:

$$\zeta^\alpha\overline{\xi}_\beta = -\frac{1}{32}\left(2\left(\overline{\xi}\Gamma_A\zeta\right)\Gamma^{A\,\alpha}{}_\beta - \frac{1}{3}\left(\overline{\xi}\Gamma_{ABC}\zeta\right)\Gamma^{ABC\,\alpha}{}_\beta + \frac{1}{120}\left(\overline{\xi}\Gamma_{ABCDE}\zeta\right)\Gamma^{ABCDE\,\alpha}{}_\beta\right).$$

terms would produce terms with $D_{11}\eta$, and supercovariance of the total supersymmetry variation of the gauginos would be spoiled.

Given the correction (4.10) to the supersymmetry variation of the gauginos, the terms that remain to be determined at this order in κ are:

(1) The correction to the supersymmetry variation of ψ_{11}; on the basis of chirality and fermi statistics, this correction can only be proportional to $(\overline{\chi}^a\Gamma_{ABC}\chi^a)\,\Gamma^{ABC}\eta$.

(2) Coefficients of all possible $\chi\chi\psi_A\psi_B$ terms in the Lagrangian; there are exactly four possible inequivalent terms of this structure. We will see momentarily that these additional four-fermi terms do appear in the Lagrangian.

We start by canceling terms $\sim \overline{\chi}\Gamma^{ABC}\chi\,\overline{\psi}_D\ldots\eta$ with D_E on one of the four fermions. If the derivative is on one of the gauginos, we can now use integration by parts to move it to either ψ_A or η. This leaves us with two classes of terms to cancel – one with $D_A\psi_B$, and one with $D_A\eta$. The $\chi\chi\eta D\psi$ terms do not get any contribution from the so far undetermined $\chi\chi\psi_A\psi_B$ terms in the Lagrangian, since at this order those will only contribute to $\chi\chi D\eta\psi$ terms. Hence, we can use cancellation of the $\chi\chi\eta D\psi$ terms to determine the correction to the supersymmetry variation of ψ_{11}; another lengthy calculation leads to[6]

$$\delta'\psi_{11} = \frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\delta(x^{11})\,(\overline{\chi}^a\Gamma^{ABC}\chi^a)\,\Gamma_{ABC}\eta. \tag{4.12}$$

Once $\delta'\psi_{11}$ has been determined, we can go on and calculate the $\overline{\chi}\Gamma_{ABC}\chi^a\,\overline{\psi}D\ldots D_E\eta$ terms; their cancellation will determine the coefficients of the remaining four-fermi terms in the Lagrangian. (As in the case of the $\chi\chi\eta D\psi$ terms, there will be a non-zero contribution from the variation of the spin connection in the gaugino kinetic term.) After some additional algebra, one obtains

$$\begin{aligned} L_4 = \frac{1}{256\pi(4\pi\kappa^2)^{2/3}}\int_{M^{10}} d^{10}x\sqrt{g}\;\overline{\chi}^a\Gamma^{ABC}\chi^a\Big(3\overline{\psi}_A\Gamma_B\psi_C - \overline{\psi}_A\Gamma_{BCD}\psi^D \\ - \frac{1}{2}\overline{\psi}_D\Gamma_{ABC}\psi^D - \frac{13}{6}\overline{\psi}^D\Gamma_{DABCE}\psi^E\Big). \end{aligned} \tag{4.13}$$

This completes the construction of the boundary Lagrangian to order $\kappa^{2/3}$, which is thus equal to the sum $L = L_0 + L_1 + L_2 + L_3 + L_4$, with the individual terms given by (4.1), (4.2), (4.6), (4.7) and (4.13).

[6] The only subtlety here is related to the cancellation of terms $\overline{\chi}^a\Gamma^{ABC}\chi^a\;\overline{\eta}\Gamma_A D_B\psi_C$, which gets a contribution from the variation of the spin connection in the gaugino kinetic term; recall the discussion of the 1.5 order formalism above.

We could stop our discussion here; instead, however, one simple point seems worth making. It turns out that the four-fermi terms that we found at order $\kappa^{2/3}$ are exactly those implied by supercovariance to this order in κ, and can therefore be absorbed into the definition of supercovariant objects. This allows us to summarize the structure of all boundary terms in the Lagrangian at order $\kappa^{2/3}$ as constructed in this section, in the following succinct formula:

$$\begin{aligned} L = \frac{1}{2\pi(4\pi\kappa^2)^{2/3}} \int_{M^{10}} d^{10}x\sqrt{g}\, \Big(& \frac{1}{4}\,\mathrm{tr}\, F_{AB}F^{AB} + \frac{1}{2}\,\mathrm{tr}\,\overline{\chi}\Gamma^A D_A(\widehat{\Omega})\chi \\ & + \frac{1}{8}\overline{\psi}_A\Gamma^{BC}\Gamma^A(F^a_{BC} + \widehat{F}^a_{BC})\chi^a + \frac{\sqrt{2}}{48}\overline{\chi}^a\Gamma^{ABC}\chi^a\, \widehat{G}_{ABC\,11}\Big)\,. \end{aligned} \tag{4.14}$$

Here the supercovariant spin connection $\widehat{\Omega}^{mn}_A$, Yang-Mills field strength $\widehat{F}^a_{AB}$, and field strength $\widehat{G}_{ABC\,11}$ are given by

$$\begin{aligned} \widehat{\Omega}_{A\,BC} &= \Omega_{A\,BC} + \frac{1}{8}\overline{\psi}^D\Gamma_{DABCE}\psi^E - \frac{1}{4}\overline{\psi}^D\Gamma_{DABC}\psi_{11}, \\ \widehat{F}^a_{AB} &= F^a_{AB} - \overline{\psi}_{[A}\Gamma_{B]}\chi^a, \\ \widehat{G}_{ABC\,11} &= G_{ABC\,11} + \frac{3\sqrt{2}}{4}\left(\overline{\psi}_{[A}\Gamma_{BC]}\psi_{11} - \overline{\psi}_{[A}\Gamma_B\psi_{C]}\right). \end{aligned} \tag{4.15}$$

(In accord with the version of the 1.5 order formalism used in this paper, the spin connection $\Omega_{A\,BC} \equiv e_{Bm}e_{Cn}\Omega^{mn}_A$ in (4.15) is a composite of $e_A{}^m$ and ψ_A, and solves the bulk equations of motion.)

Hence, we see that – just as in the case of pure eleven-dimensional supergravity [8] – no further four-fermi terms are generated at order $\kappa^{2/3}$ beyond those required by eleven-dimensional supercovariance. Of course, at higher orders in κ we encounter additional four-fermi terms that are not explained in this way – the first example of such terms is the term L_χ of (4.4), which is quartic in the gauginos and appears at relative order $\kappa^{4/3}$.

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Cosmological Solutions of Hořava-Witten Theory

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Abstract

We discuss cosmological solutions of Hořava-Witten theory describing the strongly coupled heterotic string. At energies below the grand-unified scale, the effective theory is five- not four-dimensional, where the additional coordinate parameterizes an S^1/Z_2 orbifold. Furthermore, it admits no homogeneous solutions. Rather, the static vacuum state, appropriate for a reduction to four-dimensional $N = 1$ supersymmetric models, is a BPS domain wall pair. Relevant cosmological solutions are those associated with this BPS state. In particular, such solutions must be inhomogeneous, depending on the orbifold coordinate as well as on time. We present two examples of this new type of cosmological solution, obtained by separation of variables rather than by exchange of the time and radius coordinates of a brane solution, as in previous work. The first example represents the analog of a rolling radii solution with the radii specifying the geometry of the domain wall pair. This is generalized in the second example to include a nontrivial "Ramond-Ramond" scalar.

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1 Introduction

Hořava and Witten have shown that the strongly coupled $E_8 \times E_8$ heterotic string can be identified as the eleven-dimensional limit of M-theory compactified on an S^1/Z_2 orbifold with a set of E_8 gauge fields at each ten-dimensional orbifold fixed plane [1, 2]. Furthermore, Witten has demonstrated that there exists a consistent compactification of this M-theory limit on a "deformed" Calabi-Yau threefold, leading to a supersymmetric $N = 1$ theory in four dimensions [3]. Matching at tree level to the phenomenological gravitational and grand-unified (GUT) couplings [3, 4], one finds the orbifold must be larger than the Calabi-Yau radius, by a factor of ten or so. Since the GUT scale (about 10^{16} GeV) is set by the size of the Calabi-Yau threefold, this implies that at energies below the unification scale there is a regime where the universe appears five-dimensional. This five-dimensional regime represents a new setting for early universe cosmology, which has been traditionally studied in the framework of the four-dimensional effective action.

In a previous paper [6], the effective five-dimensional Hořava-Witten theory was derived for the universal fields, which are independent of the particular form of the Calabi–Yau manifold. It has a number of interesting and unusual features. The theory lives in a five-dimensional space which is a product of a smooth four-dimensional manifold times the orbifold S^1/Z_2. As a result, it splits into a bulk $N = 1$, $d = 5$ supersymmetric theory with the gravity supermultiplet and the universal hypermultiplet, and two four-dimensional "boundary" theories which reside on the two orbifold fixed hyperplanes. The additional fields of the boundary theories are $N = 1$, $d = 4$ gauge multiplets and chiral multiplets. The reduction from eleven-dimensional supergravity to five dimensions requires the inclusion of non-zero values of the four-form field strength in the internal Calabi-Yau directions. This leads to a gauged version of five-dimensional supergravity with a potential term that had not previously been constructed. In addition, the theory has boundary potentials for the projection of the bulk scalar field onto the orbifold planes.

These potentials lead to a particularly interesting effect: the boundary sources mean that the theory has no solutions homogeneous in the orbifold direction. In particular, flat space is not a solution. Instead, the "vacuum" which leads to a supersymmetric flat four-dimensional space is a three-brane domain wall solution. The three-brane couples to the bulk potential and is supported by the sources provided by the two boundary potentials. More precisely, it is a double domain wall solution with the two $3+1$-dimensional worldvolumes each covering an orbifold plane and the orbifold itself as the transverse coordinate. It is BPS, preserving half of the $d = 5$ supersymmetries, and so is the appropriate background for a further reduction to four-dimensional $N = 1$ supergravity theories. In such a reduction, four-dimensional space-time becomes identified with the three-brane worldvolume.

Cosmologies in effective Hořava-Witten theory should thus be both *five-dimensional* and *inhomogeneous* in the extra dimension. What should realistic models look like? In the ideal case, one

would have a situation in which the internal six-dimensional Calabi-Yau space and the orbifold evolve in time for a short period and then settle down to their "phenomenological" values while the three non–compact dimensions continue to expand. Then, for late time, when all physical scales are much larger than the orbifold size, the theory is effectively four-dimensional and should, in the "static" limit, provide a realistic supergravity model of particle physics. As we have argued above, such realistic supergravity models originate from a reduction of the five-dimensional theory on its domain wall vacuum state. Hence, in the "static" limit at late time, realistic cosmological solutions should reduce to the domain wall or a perhaps a modification thereof that incorporates breaking of the remaining four-dimensional $N = 1$ supersymmetry. Consequently, one is forced to look for solutions which depend on the orbifold coordinate as well as on time. The main goal of this paper is to present simple examples of such cosmological solutions in five-dimensional heterotic M-theory to illustrate some of the characteristic cosmological features of the theory.

In earlier work [7, 8], we showed how a general class of cosmological solutions, that is, time-dependent solutions of the equations of motion that are homogeneous and isotropic in our physical $d = 3$ subspace, can be obtained in both superstring theories and M-theory defined in spacetimes *without boundary*. Loosely speaking, we showed that a cosmological solution could be obtained from any p-brane or D-brane by inverting the roles of the time and "radial" spatial coordinate. This method will clearly continue to work in Hořava-Witten theory as long as one exchanges time with a radial coordinate not aligned in the orbifold direction. An example of this in eleven-dimensions, based on the solution of [9], has been given in [10]. It can, however, not been applied to the fundamental domain wall since its radial direction coincides with the orbifold coordinate. This coordinate is bounded and cannot be turned into time. Also, as argued above, exchanging radius and time in the domain wall solution would not be desirable since it should be viewed as the vacuum state and hence should not be modified in such a way. Instead, the domain wall itself should be made time dependent thereby leading to solutions that depend on both time and the orbifold coordinate. As a result, we have to deal with coupled partial differential equations, but, under certain constraints, these can by solved by separation of variables, though the equations remain non-linear. Essentially, we are allowing the moduli describing the geometry of the domain wall and the excitations of other five-dimensional fields, to become time-dependent. Technically, we will simply take the usual Ansätze for the five-dimensional fields, but now allow the functions to depend on *both* the time and radial coordinates. We will further demand that these functions each factor into a purely time dependent piece and a purely radial dependent piece. This is not, in general, sufficient to separate the equations of motion. However, we will show that subject to certain constraints separation of variables is achieved. We can solve these separated equations and find new, cosmologically relevant solutions. In this paper, we will restrict our attention to two examples representing cosmological extensions of the pure BPS three-brane. A more general class

of solutions will be presented elsewhere [11].

The first example is simply the domain wall itself with two of its three moduli made time-dependent. We show that separation of variables occurs in this case. It turns out that these moduli behave like "rolling radii" [12] which constitute fundamental cosmological solutions in weakly coupled string theory. Unlike those rolling radii which represent scale factors of homogeneous, isotropic spaces, here they measure the separation of the two walls of the three-brane and its worldvolume size (which, at the same time, is the size of "our" three-dimensional universe). All in all, we therefore have a time-dependent domain wall pair with its shape staying rigid but its size and separation evolving like rolling radii.

For the second example, we consider a similar setting as for the first but, in addition, we allow a nonvanishing "Ramond-Ramond" scalar. This terminology is perhaps a little misleading, but relates to the fact that the scalar would be a type II Ramond-Ramond field in the case where the orbifold was replaced by a circle. This makes connection with type II cosmologies with non-trivial Ramond-Ramond fields discussed in [7, 8]. Separation of variables occurs for a specific time-independent form of this scalar. The orbifold-dependent part then coincides with the domain wall with, however, the addition of the Ramond-Ramond scalar. This non–vanishing value of the scalar breaks supersymmetry even in the static limit. We find that the time-dependent part of the equations fits into the general scheme of M-theory cosmological solutions with form fields as presented in ref. [7, 8]. Applying the results of these papers, the domain wall moduli are found to behave like rolling radii asymptotically for early and late times. The evolution rates in these asymptotic regions are different and the transitions between them can be attributed to the nontrivial Ramond-Ramond scalar.

Let us now summarize our conventions. We use coordinates x^α with indices $\alpha, \beta, \gamma, \cdots = 0, \cdots, 3, 11$ to parameterize the five-dimensional space M_5. Throughout this paper, when we refer to the orbifold, we will work in the "upstairs" picture with the orbifold S^1/Z_2 in the x^{11}-direction. We choose the range $x^{11} \in [-\pi\rho, \pi\rho]$ with the endpoints being identified. The Z_2 orbifold symmetry acts as $x^{11} \to -x^{11}$. Then there exist two four-dimensional hyperplanes fixed under the Z_2 symmetry which we denote by $M_4^{(i)}$, $i = 1, 2$. Locally, they are specified by the conditions $x^{11} = 0, \pi\rho$. The indices $\mu, \nu, \rho, \cdots = 0, \cdots, 3$ are used for the four-dimensional space orthogonal to the orbifold. Fields will be required to have a definite behaviour under the Z_2 orbifold symmetry, so that a general field Φ is either even or odd, with $\Phi(x^{11}) = \pm\Phi(-x^{11})$.

2 The five-dimensional effective action

The five-dimensional effective action for Hořava-Witten theory, obtained from the eleven-dimensional theory by compactifying on a Calabi-Yau three-fold, was derived in [6] for the universal zero modes;

that is, the five-dimensional graviton supermultiplet and the breathing mode of the Calabi-Yau space, along with its superpartners. These last fields form a hypermultiplet in five dimensions. Furthermore, the theory contains four-dimensional $N = 1$ gauge multiplets and chiral gauge matter fields on the orbifold planes. To keep the discussion as simple as possible, we do not consider the latter. This simple framework suffices for the cosmological solutions we will study in this paper. The general Lagrangian will be presented elsewhere [13, 14].

In detail, we have the five-dimensional gravity supermultiplet with the metric $g_{\alpha\beta}$ and an Abelian gauge field $\mathcal{A}_\alpha$ as the bosonic fields. The bosonic fields in the universal hypermultiplet are the real scalar field V (the dilaton, measuring the volume of the internal Calabi-Yau space), the three-form $C_{\alpha\beta\gamma}$ and the complex Ramond-Ramond scalar ξ. Note that the three-form $C_{\alpha\beta\gamma}$ can be dualized to a scalar field σ. Hence the hypermultiplet contains four real scalar fields. As explained in the introduction, all bulk fields should be even or odd under the Z_2 orbifold symmetry. One finds that the fields $g_{\mu\nu}$, $g_{11,11}$, $\mathcal{A}_{11}$, σ must be even whereas $g_{\mu 11}$, $\mathcal{A}_\mu$, ξ must be odd. If one studies cosmological solutions of the theory these transformation properties are important as they restrict the set of allowed solutions to those with the correct Z_2 symmetry. Now consider the boundary theories. In the five-dimensional space M_5, the orbifold fixed planes constitute the four-dimensional hypersurfaces $M_4^{(i)}$, $i = 1, 2$. With the standard embedding in the reduction from eleven to five dimensions, there will be an E_6 gauge field $A_\mu^{(1)}$ and gauge matter fields on the orbifold plane $M_4^{(1)}$. For simplicity, we will set these gauge matter fields to zero in the following. On the orbifold plane $M_4^{(2)}$ there is an E_8 gauge field $A_\mu^{(2)}$.

The five-dimensional effective action of Hořava-Witten theory is then given by

$$S_5 = S_{\text{bulk}} + S_{\text{bound}} \tag{2.1}$$

where

$$\begin{aligned} S_{\text{bulk}} = &-\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\left\{R + \frac{3}{2}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \frac{1}{\sqrt{2}}\epsilon^{\alpha\beta\gamma\delta\epsilon}\mathcal{A}_\alpha\mathcal{F}_{\beta\gamma}\mathcal{F}_{\delta\epsilon} + \frac{1}{2V^2}\partial_\alpha V\partial^\alpha V\right. \\ &+ \frac{1}{2V^2}\left[\partial_\alpha\sigma - i(\xi\partial_\alpha\bar{\xi} - \bar{\xi}\partial_\alpha\xi) - 2\alpha_0\epsilon(x^{11})\mathcal{A}_\alpha\right]\left[\partial^\alpha\sigma - i(\xi\partial^\alpha\bar{\xi} - \bar{\xi}\partial^\alpha\xi) - 2\alpha_0\epsilon(x^{11})\mathcal{A}^\alpha\right] \\ &\left.+ \frac{2}{V}\partial_\alpha\xi\partial^\alpha\bar{\xi} + \frac{1}{3V^2}\alpha_0^2\right\} \end{aligned} \tag{2.2}$$

$$\begin{aligned} S_{\text{bound}} = &\frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(1)}}\sqrt{-g}\,V^{-1}\alpha_0 - \frac{\sqrt{2}}{\kappa_5^2}\int_{M_4^{(2)}}\sqrt{-g}\,V^{-1}\alpha_0 \\ &- \frac{1}{16\pi\alpha_{\text{GUT}}}\sum_{i=1}^{2}\int_{M_4^{(i)}}\sqrt{-g}\left\{V\,\text{tr}F_{\mu\nu}^{(i)}F^{(i)\mu\nu} - \sigma\,\text{tr}F_{\mu\nu}^{(i)}\tilde{F}^{(i)\mu\nu}\right\}. \end{aligned} \tag{2.3}$$

where $\mathcal{F}_{\alpha\beta} = \partial_\alpha\mathcal{A}_\beta - \partial_\beta\mathcal{A}_\alpha$ and the $F_{\mu\nu}^{(i)}$ are the field strengths of the boundary gauge fields, while $\tilde{F}^{(i)\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}^{(i)}$. Furthermore, κ_5 and α_{GUT} are the five-dimensional Newton constant and the gauge coupling respectively. The constant α_0 in the above action can be computed for

a given internal Calabi-Yau space. Explicit formulae are presented in ref. [3, 6]. In the above expression, we have dropped higher-derivative terms. The sigma-model for the scalar fields is the well-known coset $\mathcal{M}_Q = SU(2,1)/SU(2) \times U(1)$ of the universal hypermultiplet. The coupling of σ to $\mathcal{A}_\alpha$ implies that a $U(1)$ symmetry on $\mathcal{M}_Q$ has been gauged. This gauging also induces the α_0-dependent potential term in (2.3). It has been demonstrated [6, 14] that the above action is indeed the bosonic part of a minimal $N = 1$ gauged supergravity theory in five dimensions coupled to chiral boundary theories.

The most striking features of this action from the viewpoint of cosmology (and otherwise) are the bulk and boundary potentials for the dilaton V in S_{bulk} and S_{bound}. These potential terms are proportional to the parameter α_0 and their origin is directly related to the nonzero internal four-form that had to be included in the dimensional reduction from eleven dimensions. The boundary potentials lead to sources in the Einstein equation and the equation of motion for V and σ that are proportional to $\delta(x^{11})$ or $\delta(x^{11} - \pi\rho)$. Hence, as long as V is finite (the internal Calabi-Yau space is compact) purely time-dependent solutions of the theory do not exist as they could never cancel these delta-function sources. One is therefore led to always consider dependence on time and the orbifold coordinate x^{11}. The presence of a bulk potential proportional to V^{-2} seems to indicate that the dilaton has a runaway behaviour and the internal space decompactifies at late time. This picture, however, is too naive in that it ignores the boundary potentials and the Z_2 symmetries of the fields. In fact, as we will show, the correct static domain wall vacuum of the theory depends on the orbifold direction in a way so as to exactly cancel these potentials. Consequently, it is important to note that for cosmological solutions based on the domain wall the time-dependent scale factors do not feel the potential terms.

3 The domain-wall vacuum solution

In this section, we would like to review the static "vacuum" solution of the five-dimensional Hořava-Witten theory, as given in [6]. As argued in the introduction, this solution is the basis for physically relevant cosmological solutions. It is clear from the five-dimensional action given in the previous section, that flat spacetime is not a solution of the equations of motion. It is precluded from being a solution by the potential terms, both in the bulk and on the boundaries. If not flat space, what is the natural vacuum solution? To answer this, notice that the theory (2.1) has all of the prerequisites necessary for a three-brane solution to exist. Generally, in order to have a $(D-2)$-brane in a D-dimensional theory, one needs to have a $(D-1)$-form field or, equivalently, a cosmological constant. This is familiar from the eight-brane [15] in the massive type IIA supergravity in ten dimensions [16], and has been systematically studied for theories in arbitrary dimension obtained by generalized (Scherk-Schwarz) dimensional reduction [17]. In our case, this cosmological term is

provided by the bulk potential term in the action (2.1), precisely the term that disallowed flat space as a solution. From the viewpoint of the bulk theory, we could have multi three-brane solutions with an arbitrary number of parallel branes located at various places in the x^{11} direction. However, elementary brane solutions have singularities at the location of the branes, needing to be supported by source terms. The natural candidates for those source terms, in our case, are the boundary actions. This restricts the possible solutions to those representing a pair of parallel three-branes corresponding to the orbifold planes. This pair of domain walls can be viewed as the "vacuum" of the five-dimensional theory, in the sense that it provides the appropriate background for a reduction to the $d = 4$, $N = 1$ effective theory.

From the above discussion, it is clear that in order to find a three-brane solution, we should start with the Ansatz

$$\begin{aligned} ds_5^2 &= a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \\ V &= V(y) \end{aligned} \tag{3.1}$$

where a and b are functions of $y = x^{11}$ and all other fields vanish. The general solution for this Ansatz, satisfying the equations of motion derived from action (2.1), is given by

$$\begin{aligned} a &= a_0 H^{1/2} \\ b &= b_0 H^2 \qquad\qquad H = \frac{\sqrt{2}}{3}\alpha_0 |y| + h_0 \\ V &= b_0 H^3 \end{aligned} \tag{3.2}$$

where a_0, b_0 and h_0 are constants. We note that the boundary source terms have fixed the form of the harmonic function H in the above solution. Without specific information about the sources, the function H would generically be glued together from an arbitrary number of linear pieces with slopes $\pm\sqrt{2}\alpha_0/3$. The edges of each piece would then indicate the location of the source terms. The necessity of matching the boundary sources at $y = 0$ and $\pi\rho$, however, has forced us to consider only two such linear pieces, namely $y \in [0, \pi\rho]$ and $y \in [-\pi\rho, 0]$. These pieces are glued together at $y = 0$ and $\pi\rho$ (recall here that we have identified $\pi\rho$ and $-\pi\rho$). To see this explicitly, let us consider one of the equations of motion; specifically, the equation derived from the variation of $g_{\mu\nu}$. For the Ansatz in (3.1), this is given by

$$\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{12}\frac{V'^2}{V^2} + \frac{\alpha_0^2}{18}\frac{b^2}{V^2} = \frac{\sqrt{2}\alpha_0}{3}\frac{b}{V}\left(\delta(y) - \delta(y - \pi\rho)\right) \tag{3.3}$$

where the prime denotes differentiation with respect to y. The term involving the delta functions arises from the stress energy on the boundary planes. Inserting the solution (3.2) in this equation, we have

$$\partial_y^2 H = \frac{2\sqrt{2}}{3}\alpha_0(\delta(y) - \delta(y - \pi\rho)) \tag{3.4}$$

which shows that the solution represents two parallel three-branes located at the orbifold planes. Using the five-dimensional supersymmetry transformations presented in ref. [6], one can check that this solution indeed preserves four of the eight supersymmetries of the theory.

Let us discuss the meaning of this solution. As is apparent from the Ansatz (3.1), it has $3+1$-dimensional Poincaré invariance and, as just stated, it preserves four supercharges. Therefore, a dimensional reduction to four dimensions on this solution leads to an $N=1$ supergravity theory. In fact, this is just the "physical" four-dimensional effective theory of strongly coupled heterotic string theory which is the starting point of low energy particle phenomenology. This has been explicitly demonstrated in ref. [13, 14]. The two parallel three-branes of the solution, separated by the bulk, are oriented in the four uncompactified space-time dimensions, and carry the physical low-energy gauge and matter fields. Therefore, from the low-energy point of view where the orbifold is not resolved, the three-brane worldvolume is identified with four-dimensional space-time. In this sense the Universe lives on the worldvolume of a three-brane. It is the purpose of the following sections to put this picture into the context of cosmology; that is, to make it dynamical. Consequently, we are looking for time dependent solutions based on the static domain wall which we have just presented.

4 The domain-wall cosmological solution

In this section, we will present a cosmological solution related to the static domain wall vacuum of the previous section. As discussed in ref. [7, 8], a convenient way to find such a solution is to use Ansatz (3.1) where the $y = x^{11}$ coordinate in the functions a, b and V is replaced by the time coordinate τ. However, in Hořava-Witten theory the boundary planes preclude this from being a solution of the equations of motion, since it does not admit homogeneous solutions. To see this explicitly, let us consider the g_{00} equation of motion, where we replace $a(y) \to \alpha(\tau), b(y) \to \beta(\tau)$ and $V \to \gamma(\tau)$. We find that

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} - \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} - \frac{\alpha_0^2}{18}\frac{1}{\gamma^2} = -\frac{\sqrt{2}\alpha_0}{3}\frac{1}{\beta\gamma}\left(\delta(y) - \delta(y-\pi\rho)\right) \ , \tag{4.1}$$

where the dot denotes differentiation with respect to τ. Again, the term containing the delta functions arises from the boundary planes. It is clear that, because of the y-dependence introduced by the delta functions, this equation has no globally defined solution. The structure of equation (4.1) suggests that a solution might be found if one were to let functions a, b and V depend on both τ and y coordinates. This would be acceptable from the point of view of cosmology, since any such solution would be homogeneous and isotropic in the spatial coordinates x^m where $m, n, r, \cdots = 1, 2, 3$. In fact, the previous Ansatz was too homogeneous, being independent of the y coordinate as well. Instead, we are interested in solutions where the inhomogeneous vacuum domain wall evolves in time.

We now construct a cosmological solution where all functions depend on both τ and y. We start with the Ansatz

$$\begin{aligned} ds_5^2 &= -N(\tau,y)^2 d\tau^2 + a(\tau,y)^2 dx^m dx^n \eta_{mn} + b(\tau,y)^2 dy^2 \\ V &= V(\tau,y) \end{aligned} \tag{4.2}$$

Note that we have introduced a separate function N into the purely temporal part of the metric. This Ansatz leads to equations of motion that mix the τ and y variables in a complicated non-linear way. In order to solve this system of equations, we will try to separate the two variables. That is, we let

$$\begin{aligned} N(\tau,y) &= n(\tau)a(y) \\ a(\tau,y) &= \alpha(\tau)a(y) \\ b(\tau,y) &= \beta(\tau)b(y) \\ V(\tau,y) &= \gamma(\tau)V(y) \end{aligned} \tag{4.3}$$

There are two properties of this Ansatz that we wish to point out. The first is that for $n = \alpha = \beta = \gamma = 1$ it becomes identical to (3.1). Secondly, we note that n can be chosen to be any function by performing a redefinition of the τ variable. That is, we can think of n as being subject to a gauge transformation. There is no a priori reason to believe that separation of variables will lead to a solution of the equations of motion derived from the action (2.1). However, as we now show, there is indeed such a solution. It is instructive to present one of the equations of motion. With the above Ansatz, the g_{00} equation of motion is given by [1]

$$\frac{a^2}{b^2}\left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{12}\frac{V'^2}{V^2} + \frac{\alpha_0^2}{18}\frac{b^2}{V^2}\frac{\beta^2}{\gamma^2} - \frac{\sqrt{2}}{3}\alpha_0\frac{b}{V}(\delta(y)-\delta(y-\pi\rho))\frac{\beta}{\gamma}\right) = \frac{\beta^2}{n^2}\left(\frac{\dot\alpha^2}{\alpha^2} + \frac{\dot\alpha}{\alpha}\frac{\dot\beta}{\beta} - \frac{1}{12}\frac{\dot\gamma^2}{\gamma^2}\right) \tag{4.4}$$

Note that if we set $n = \alpha = \beta = \gamma = 1$ this equation becomes identical to (3.3). Similarly, if we set $a = b = V = 1$ and take the gauge $n = 1$ this equation becomes the same as (4.1). As is, the above equation does not separate. However, the obstruction to a separation of variables is the two terms proportional to α_0. Note that both of these terms would be strictly functions of y only if we demanded that $\beta \propto \gamma$. Without loss of generality, one can take

$$\beta = \gamma \, . \tag{4.5}$$

We will, henceforth, assume this is the case. Note that this result is already indicated by the structure of integration constants (moduli) in the static domain wall solution (3.2). With this

[1] From now on, we denote by a, b, V the y-dependent part of the Ansatz (4.3).

condition, the left hand side of equation (4.4) is purely y dependent, whereas the right hand side is purely τ dependent. Both sides must now equal the same constant which, for simplicity, we take to be zero. The equation obtained by setting the left hand side to zero is identical to the pure y equation (3.3). The equation for the pure τ dependent functions is

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} - \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} = 0 \tag{4.6}$$

Hence, separation of variables can be achieved for the g_{00} equation by demanding that (4.5) is true. What is more remarkable is that, subject to the constraint that $\beta = \gamma$, all the equations of motion separate. The pure y equations are identical to those of the previous section and, hence, the domain wall solution (3.2) remains valid as the y–dependant part of the solution.

The full set of τ equations is found to be

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} - \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} = 0 \tag{4.7}$$

$$2\frac{\ddot{\alpha}}{\alpha} - 2\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n} + \frac{\ddot{\beta}}{\beta} - \frac{\dot{\beta}}{\beta}\frac{\dot{n}}{n} + \frac{\dot{\alpha}^2}{\alpha^2} + 2\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} + \frac{1}{4}\frac{\dot{\gamma}^2}{\gamma^2} = 0 \tag{4.8}$$

$$\frac{\ddot{\alpha}}{\alpha} - \frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n} + \frac{\dot{\alpha}^2}{\alpha^2} + \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} = 0 \tag{4.9}$$

$$\frac{\ddot{\gamma}}{\gamma} + 3\frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma} + \frac{\dot{\beta}\dot{\gamma}}{\beta\gamma} - \frac{\dot{\gamma}^2}{\gamma^2} - \frac{\dot{n}\dot{\gamma}}{n\gamma} = 0 \tag{4.10}$$

In these equations we have displayed β and γ independently, for reasons to become apparent shortly. Of course, one must solve these equations subject to the condition that $\beta = \gamma$. As a first attempt to solve these equations, it is most convenient to choose a gauge for which

$$n = \text{const} \tag{4.11}$$

so that τ becomes proportional to the comoving time t, since $dt = n(\tau)d\tau$. In such a gauge, the equations simplify considerably and we obtain the solution

$$\begin{aligned} \alpha &= A|t - t_0|^p \\ \beta &= \gamma = B|t - t_0|^q \end{aligned} \tag{4.12}$$

where

$$p = \frac{3}{11}(1 \mp \frac{4}{3\sqrt{3}}) \,, \qquad q = \frac{2}{11}(1 \pm 2\sqrt{3}) \tag{4.13}$$

and A, B and t_0 are arbitrary constants. We have therefore found a cosmological solution, based on the separation Ansatz (4.3), with the y-dependent part being identical to the domain wall solution (3.2) and the scale factors α, β, γ evolving according to the power laws (4.12). This means that the shape of the domain wall pair stays rigid while its size and the separation between the walls evolve in time. Specifically, α measures the size of the spatial domain wall worldvolume (the size of the three-dimensional universe), while β specifies the separation of the two walls (the size of the orbifold). Due to the separation constraint $\gamma = \beta$, the time evolution of the Calabi-Yau volume, specified by γ, is always tracking the orbifold. From this point of view, we are allowing two of the three moduli in (3.2), namely a_0 and b_0, to become time-dependent. Since these moduli multiply the harmonic function H, it is then easy to see why a solution by separation of variables was appropriate.

To understand the structure of the above solution, it is useful to rewrite its time dependent part in a more systematic way using the formalism developed in ref. [7, 8]. First, let us define new functions $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ by

$$\alpha = e^{\hat{\alpha}}, \qquad \beta = e^{\hat{\beta}}, \qquad \gamma = e^{6\hat{\gamma}} \tag{4.14}$$

and introduce the vector notation

$$\vec{\alpha} = (\alpha^i) = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix} , \qquad \vec{d} = (d_i) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} . \tag{4.15}$$

Note that the vector $\vec{d}$ specifies the dimensions of the various subspaces, where the entry $d_1 = 3$ is the spatial worldvolume dimensions, $d_2 = 1$ is the orbifold dimension and we insert 0 for the dilaton. On the "moduli space" spanned by $\vec{\alpha}$ we introduce the metric

$$\begin{aligned} G_{ij} &= 2(d_i\delta_{ij} - d_i d_j) \\ G_{in} &= G_{ni} = 0 \\ G_{nn} &= 36 \, , \end{aligned} \tag{4.16}$$

which in our case explicitly reads

$$G = -12 \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} . \tag{4.17}$$

Furthermore, we define E by

$$E = \frac{e^{\vec{d}\cdot\vec{\alpha}}}{n} = \frac{e^{3\hat{\alpha}+\hat{\beta}}}{n} \, . \tag{4.18}$$

The equations of motion (4.7)-(4.10) can then be rewritten as

$$\frac{1}{2}E\dot{\vec{\alpha}}^T G\dot{\vec{\alpha}} = 0\,, \qquad \frac{d}{d\tau}\left(EG\dot{\vec{\alpha}}\right) = 0\,. \tag{4.19}$$

It is straightforward to show that if we choose a gauge $n = \text{const}$, these two equations exactly reproduce the solution given in (4.12) and (4.13). The importance of this reformulation of the equations of motion lies, however, in the fact that we now get solutions more easily by exploiting the gauge choice for n. For example, let us now choose the gauge

$$n = e^{\vec{d}\cdot\vec{\alpha}}\,. \tag{4.20}$$

Note that in this gauge $E = 1$. The reader can verify that this gauge choice greatly simplifies solving the equations. The result is that

$$\begin{aligned} \hat{\alpha} &= 6\hat{\gamma} = C\tau + k_1 \\ \hat{\beta} &= (6 \pm 4\sqrt{3})C\tau + k_2 \end{aligned} \tag{4.21}$$

where C, k_1 and k_2 are arbitrary constants. Of course this solution is completely equivalent to the previous one, eq. (4.12), but written in a different gauge. We will exploit this gauge freedom to effect in the next section.

To discuss cosmological properties we define the Hubble parameters

$$\vec{H} = \frac{d}{dt}\vec{\alpha} \tag{4.22}$$

where t is the comoving time. From (4.12) and (4.14) we easily find

$$\vec{H} = \frac{\vec{p}}{t - t_0}\,, \qquad \vec{p} = \begin{pmatrix} p \\ q \\ \frac{1}{6}q \end{pmatrix}\,. \tag{4.23}$$

Note that the powers $\vec{p}$ satisfy the constraints

$$\vec{p}^T G\vec{p} = 0 \qquad \vec{d}\cdot\vec{p} = 1\,. \tag{4.24}$$

These relations are characteristic for rolling radii solutions [12] which are fundamental cosmological solutions of weakly coupled heterotic string theory. Comparison of the equations of motion (4.19) indeed shows that the scale factors $\vec{\alpha}$ behave like rolling radii. The original rolling radii solutions describe freely evolving scale factors of a product of homogeneous, isotropic spaces. In our case, the scale factors also evolve freely (since the time-dependent part of the equations of motion, obtained after separating variables, does not contain a potential) but they describe the time evolution of the domain wall. This also proves our earlier claim that the potential terms in the five-dimensional

action (2.1) do not directly influence the time-dependence but are canceled by the static domain wall part of the solution.

Let us now be more specific about the cosmological properties of our solution. First note from eq. (4.23) that there exist two different types of time ranges, namely $t < t_0$ and $t > t_0$. In the first case, which we call the $(-)$ branch, the evolution starts at $t \to -\infty$ and runs into a future curvature singularity [7, 8] at $t = t_0$. In the second case, called the $(+)$ branch, we start out in a past curvature singularity at $t = t_0$ and evolve toward $t \to \infty$. In summary, we therefore have the branches

$$t \in \begin{cases} [-\infty, t_0] & (-) \text{ branch} \\ [t_0, +\infty] & (+) \text{ branch} \end{cases} . \tag{4.25}$$

For both of these branches we have two options for the powers $\vec{p}$, defined in eq. (4.23), corresponding to the two different signs in eq. (4.13). Numerically, we find

$$\vec{p}_\uparrow \simeq \begin{pmatrix} +.06 \\ +.81 \\ +.14 \end{pmatrix} , \qquad \vec{p}_\downarrow \simeq \begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix} \tag{4.26}$$

for the upper and lower sign in (4.13) respectively. We recall that the three entries in these vectors specify the evolution powers for the spatial worldvolume of the three-brane, the domain wall separation and the Calabi-Yau volume. The expansion of the domain wall worldvolume has so far been measured in terms of the five-dimensional Einstein frame metric $g^{(5)}_{\mu\nu}$. This is also what the above numbers p_1 reflect. Alternatively, one could measure this expansion with the four-dimensional Einstein frame metric $g^{(4)}_{\mu\nu}$ so that the curvature scalar on the worldvolume is canonically normalized. From the relation

$$g^{(4)}_{\mu\nu} = (g_{11,11})^{1/2}\, g^{(5)}_{\mu\nu} \tag{4.27}$$

we find that this modifies p_1 to

$$\tilde{p}_1 = p_1 + \frac{p_2}{2} . \tag{4.28}$$

In the following, we will discuss both frames. We recall that the separation condition $\beta = \gamma$ implies that the internal Calabi-Yau space always tracks the orbifold. In the discussion we can, therefore, concentrate on the spatial worldvolume and the orbifold, corresponding to the first and second entries in (4.26). Let us first consider the $(-)$ branch. In this branch $t \in [-\infty, t_0]$ and, hence, $t - t_0$ is always negative. It follows from eq. (4.23) that a subspace will expand if its $\vec{p}$ component is negative and contract if it is positive. For the first set of powers $\vec{p}_\uparrow$ in eq. (4.26) both the worldvolume and the orbifold contract in the five-dimensional Einstein frame. The same conclusion holds in the four-dimensional Einstein frame. For the second set, $\vec{p}_\downarrow$, in both frames

the worldvolume contracts while the orbifold expands. Furthermore, since the Hubble parameter of the orbifold increases in time the orbifold undergoes superinflation.

Now we turn to the (+) branch. In this branch $t \in [t_0, \infty]$ and, hence, $t - t_0$ is always positive. Consequently, a subspace expands for a positive component of $\vec{p}$ and contracts otherwise. In addition, since the absolute values of all powers $\vec{p}$ are smaller than one, an expansion is always subluminal. For the vector $\vec{p}_\uparrow$ the worldvolume and the orbifold expand in both frames. On the other hand, the vector $\vec{p}_\downarrow$ describes an expanding worldvolume and a contracting orbifold in both frames. This last solution perhaps corresponds most closely to our notion of the early universe.

5 Cosmological solutions with Ramond forms

Thus far, we have looked for both static and cosmological solutions where the form fields $\xi, \mathcal{A}_\alpha$ and σ have been set to zero. As discussed in previous papers [7, 8], turning on one or several such fields can drastically alter the solutions and their cosmological properties. Hence, we would like to explore cosmological solutions with such non-trivial fields. For clarity, in this paper we will restrict the discussion to turning on the Ramond-Ramond scalar ξ only, postponing the general discussion to another publication [11].

The Ansatz we will use is the following. For the metric and dilaton field, we choose

$$\begin{aligned} ds_5^2 &= -N(\tau, y)^2 d\tau^2 + a(\tau, y)^2 dx^m dx^n \eta_{mn} + b(\tau, y)^2 dy^2 \\ V &= V(\tau, y) \ . \end{aligned} \tag{5.1}$$

For the ξ field, we assume that $\xi = \xi(\tau, y)$ and, hence, the field strength $F_\alpha = \partial_\alpha \xi$ is given by

$$F_0 = Y(\tau, y) \ , \qquad F_5 = X(\tau, y) \ . \tag{5.2}$$

All other components of F_α vanish. Note that since ξ is complex, both X and Y are complex. Once again, we will solve the equations of motion by separation of variables. That is, we let

$$\begin{aligned} N(\tau, y) &= n(\tau) N(y) \\ a(\tau, y) &= \alpha(\tau) a(y) \\ b(\tau, y) &= \beta(\tau) b(y) \\ V(\tau, y) &= \gamma(\tau) V(y) \end{aligned} \tag{5.3}$$

and

$$X(\tau, y) = \chi(\tau) X(y) \tag{5.4}$$

$$Y(\tau, y) = \phi(\tau) Y(y) \ . \tag{5.5}$$

Note that, in addition to the ξ field, we have also allowed for the possibility that $N(y) \neq a(y)$. Again, there is no a priori reason to believe that a solution can be found by separation of variables. However, as above, there is indeed such a solution, although the constraints required to separate variables are more subtle. It is instructive to present one of the equations of motion. With the above Ansatz, the g_{00} equation of motion becomes [2]

$$\frac{N^2}{b^2}\left(\frac{a''}{a}+\frac{a'^2}{a^2}-\frac{a'}{a}\frac{b'}{b}+\frac{1}{12}\frac{V'^2}{V^2}+\frac{\alpha_0^2}{18}\frac{b^2}{V^2}\frac{\beta^2}{\gamma^2}-\frac{\sqrt{2}\alpha_0}{3}\frac{b}{V}(\delta(y)-\delta(y-\pi\rho))\frac{\beta}{\gamma}\right)= \\ \frac{\beta^2}{n^2}\left(\frac{\dot\alpha^2}{\alpha^2}+\frac{\dot\alpha}{\alpha}\frac{\dot\beta}{\beta}-\frac{1}{12}\frac{\dot\gamma^2}{\gamma^2}\right)-\frac{N^2}{3b^2}\frac{|X|^2}{V}\frac{|\chi|^2}{\gamma}-\frac{\beta^2}{3n^2}\frac{|Y|^2}{V}\frac{|\phi|^2}{\gamma} \tag{5.6}$$

Note that if we set $X = Y = 0$ and $N = a$ this equation becomes identical to (4.4). We now see that there are two different types of obstructions to the separation of variables. The first type, which we encountered in the previous section, is in the two terms proportional to α_0. Clearly, we can separate variables only if we demand that

$$\beta = \gamma \tag{5.7}$$

as we did previously. However, for non-vanishing X and Y this is not sufficient. The problem, of course, comes from the last two terms in (5.6). There are a number of options one could try in order to separate variables in these terms. It is important to note that X and Y are not completely independent, but are related to each other by the integrability condition $\partial_\tau X(\tau, y) = \partial_y Y(\tau, y)$. We find that, because of this condition, it is impossible to obtain a solution by separation of variables that has both $X(\tau, y)$ and $Y(\tau, y)$ non-vanishing. Now $X(\tau, y)$, but not $Y(\tau, y)$, can be made to vanish by taking $\xi = \xi(\tau)$; that is, ξ is a function of τ only. However we can find no solution by separation of variables under this circumstance. Thus, we are finally led to the choice $\xi = \xi(y)$. In this case $Y(\tau, y) = 0$ and we can, without lose of generality, choose

$$\chi = 1\ . \tag{5.8}$$

At this point, the only obstruction to separation of variables in equation (5.6) is the next to last term, $N^2|X|^2/3b^2V\gamma$. Setting $\gamma = \text{const}$ is too restrictive, so we must demand that

$$X = \frac{bV^{1/2}}{N}c_0 e^{i\theta(y)} \tag{5.9}$$

where c_0 is a non-zero but otherwise arbitrary real constant and $\theta(y)$ is an, as yet, undetermined phase. Putting this condition into the ξ equation of motion

$$\partial_y\left(\frac{a^3N}{bV}X\right) = 0 \tag{5.10}$$

[2] In the following, N, a, b, V denote the y-dependent part of the Ansatz (5.3).

we find that θ is a constant θ_0 and $a \propto V^{\frac{1}{6}}$ with arbitrary coefficient. Note that the last condition is consistent with the static vacuum solution (3.2). Inserting this result into the g_{05} equation of motion

$$\frac{\dot{\alpha}}{\alpha}\left(\frac{a'}{a} - \frac{N'}{N}\right) = \frac{\dot{\beta}}{\beta}\left(\frac{a'}{a} - \frac{1}{6}\frac{V'}{V}\right) \tag{5.11}$$

we learn that $N \propto a$ with arbitrary coefficient. Henceforth, we choose $N = a$ which is consistent with the static vacuum solution (3.2). Inserting all of these results, the g_{00} equation of motion now becomes

$$\frac{a^2}{b^2}\left(\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a}\frac{b'}{b} + \frac{1}{12}\frac{V'^2}{V^2} + \frac{\alpha_0^2}{18}\frac{b^2}{V^2} - \frac{\sqrt{2}}{3}\alpha_0\frac{b}{V}(\delta(y) - \delta(y - \pi\rho))\right) = \frac{\beta^2}{n^2}\left(\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} - \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2}\right) - \frac{c_0^2}{3}\frac{1}{\gamma} \tag{5.12}$$

Note that the left hand side is of the same form as the static vacuum equation (3.3). The effect of turning on the ξ background is to add a purely τ dependent piece to the right hand side. Putting these results into the remaining four equations of motion, we find that they too separate, with the left hand side being purely y dependent and the right hand side purely τ dependent. Again, we find that in these equations the left hand sides are identical to those in the static vacuum equations and the effect of turning on ξ is to add extra τ dependent terms to the right hand sides. In each equation, both sides must now equal the same constant which, for simplicity, we take to be zero. The y equations for a, b and V thus obtained by setting the left hand side to zero are identical to the static vacuum equations. Hence, we have shown that

$$\begin{aligned} N = a &= a_0 H^{1/2} \\ b &= b_0 H^2 \qquad\qquad H = \frac{\sqrt{2}}{3}\alpha_0|y| + h_0 \\ V &= b_0 H^3 \\ X &= x_0 H^3 \end{aligned} \tag{5.13}$$

where $x_0 = c_0 e^{i\theta_0} a_0^{-1} b_0^{3/2}$ is an arbitrary constant.

The τ equations obtained by setting the right hand side to zero are the following.

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} - \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} - \frac{c_0^2}{3}\frac{n^2}{\beta^2\gamma} = 0 \tag{5.14}$$

$$2\frac{\ddot{\alpha}}{\alpha} - 2\frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n} + \frac{\ddot{\beta}}{\beta} - \frac{\dot{\beta}}{\beta}\frac{\dot{n}}{n} + \frac{\dot{\alpha}^2}{\alpha^2} + 2\frac{\dot{\alpha}}{\alpha}\frac{\dot{\beta}}{\beta} + \frac{1}{4}\frac{\dot{\gamma}^2}{\gamma^2} - c_0^2\frac{n^2}{\beta^2\gamma} = 0 \tag{5.15}$$

$$\frac{\ddot{\alpha}}{\alpha} - \frac{\dot{\alpha}}{\alpha}\frac{\dot{n}}{n} + \frac{\dot{\alpha}^2}{\alpha^2} + \frac{1}{12}\frac{\dot{\gamma}^2}{\gamma^2} + \frac{c_0^2}{3}\frac{n^2}{\beta^2\gamma} = 0 \tag{5.16}$$

$$\frac{\ddot{\gamma}}{\gamma} + 3\frac{\dot{\alpha}\dot{\gamma}}{\alpha\gamma} + \frac{\dot{\beta}\dot{\gamma}}{\beta\gamma} - \frac{\dot{\gamma}^2}{\gamma^2} - \frac{\dot{n}\dot{\gamma}}{n\gamma} - 2c_0^2\frac{n^2}{\beta^2\dot{\gamma}} = 0 \tag{5.17}$$

In these equations we have, once again, displayed β and γ independently, although they should be solved subject to the condition $\beta = \gamma$. Note that the above equations are similar to the τ equations in the previous section, but each now has an additional term proportional to c_0^2. These extra terms considerably complicate finding a solution of the τ equations. Here, however, is where the formalism introduced in the previous section becomes important. Defining $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ as in (4.14), and $\vec{\alpha}, E$ and G as in (4.15), (4.18) and (4.17) respectively, the equations (5.14)-(5.17) can be written in the form

$$\frac{1}{2}E\dot{\vec{\alpha}}^T G\dot{\vec{\alpha}} + E^{-1}U = 0\ , \qquad \frac{d}{d\tau}\left(EG\dot{\vec{\alpha}}\right) + E^{-1}\frac{\partial U}{\partial\vec{\alpha}} = 0 \tag{5.18}$$

where the potential U is defined as

$$U = 2c_0^2 e^{\vec{q}\cdot\vec{\alpha}} \tag{5.19}$$

with

$$\vec{q} = \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}\ . \tag{5.20}$$

We can now exploit the gauge freedom of n to simplify these equations. Choose the gauge

$$n = e^{(\vec{d}-\vec{q})\cdot\vec{\alpha}} \tag{5.21}$$

where $\vec{d}$ is defined in (4.15). Then E becomes proportional to the potential U so that the potential terms in (5.18) turn into constants. Thanks to this simplification, the equations of motion can be integrated which leads to the general solution [7, 8]

$$\vec{\alpha} = \vec{c}\,\ln|\tau_1 - \tau| + \vec{w}\,\ln\left(\frac{s\tau}{\tau_1 - \tau}\right) + \vec{k} \tag{5.22}$$

where τ_1 is an arbitrary parameter which we take, without loss of generality, to be positive and

$$\vec{c} = 2\frac{G^{-1}\vec{q}}{<\vec{q},\vec{q}>}\ , \qquad s = \mathrm{sign}(<\vec{q},\vec{q}>)\ . \tag{5.23}$$

The scalar product is defined as $<\vec{q},\vec{q}>=\vec{q}^T G^{-1}\vec{q}$. The vectors $\vec{w}$ and $\vec{k}$ are integration constants subject to the constraints

$$\begin{aligned} \vec{q}\cdot\vec{w} &= 1 \\ \vec{w}^T G\vec{w} &= 0 \\ \vec{q}\cdot\vec{k} &= \ln\left(c_0^2|<\vec{q},\vec{q}>|\right) \end{aligned} \tag{5.24}$$

This solution is quite general in that it describes an arbitrary number of scale factors with equations of motion given by (5.18). Let us now specify to our example. For G and $\vec{q}$ as given in eq. (4.17) and (5.20) we find that

$$< \vec{q}, \vec{q} >= 1 \tag{5.25}$$

hence $s = 1$, and

$$\vec{c} = \begin{pmatrix} 0 \\ -2 \\ -\frac{1}{3} \end{pmatrix} . \tag{5.26}$$

Recall that we must, in addition, demand that $\beta = \gamma$. Note that the last two components of $\vec{c}$ are consistent with this equality. We can also solve the constraints (5.24) subject to the condition $\beta = \gamma$. The result is

$$\vec{w} = \begin{pmatrix} w_3 + \frac{1}{6} \\ 6w_3 \\ w_3 \end{pmatrix} , \qquad \vec{k} = \begin{pmatrix} k_3 + \frac{1}{6}\ln c_0^2 \\ 6k_3 \\ k_3 \end{pmatrix} \tag{5.27}$$

where

$$w_3 = -\frac{1}{6} \pm \frac{\sqrt{3}}{12} \tag{5.28}$$

and k_3 is arbitrary. We conclude that in the gauge specified by (5.21), the solution is given by

$$\begin{aligned} \hat{\alpha} &= (w_3 + \frac{1}{6}) \ln\left(\frac{\tau}{\tau_1 - \tau}\right) + k_3 + \frac{1}{6}\ln c_0^2 \\ \hat{\beta} &= -2\ln|\tau_1 - \tau| + 6w_3 \ln\left(\frac{\tau}{\tau_1 - \tau}\right) + 6k_3 \\ \hat{\gamma} &= -\frac{1}{3}\ln|\tau_1 - \tau| + w_3 \ln\left(\frac{\tau}{\tau_1 - \tau}\right) + k_3 \end{aligned} \tag{5.29}$$

with w_3 as above. As a consequence of $s = 1$, the range for τ is restricted to

$$0 < \tau < \tau_1 \tag{5.30}$$

in this solution. Let us now summarize our result. We have found a cosmological solution with a nontrivial Ramond-Ramond scalar ξ starting with the separation Ansatz (5.3). To achieve separation of variables we had to demand that $\beta = \gamma$, as previously, and that the Ramond-Ramond scalar depends on the orbifold coordinate but not on time. Then the orbifold dependent part of the solution is given by eq. (5.13) and is identical to the static domain wall solution with the addition of the Ramond-Ramond scalar. The time dependent part, in the gauge (5.21), is specified

by eq. (5.29). Furthermore, we have found that the time-dependent part of the equations of motion can be cast in a form familiar from cosmological solutions studied previously [7, 8]. Those solutions describe the evolution for scale factors of homogeneous, isotropic subspaces in the presence of antisymmetric tensor fields and are, therefore, natural generalizations of the rolling radii solutions. Each antisymmetric tensor field introduces an exponential type potential similar to the one in eq. (5.19). For the case with only one nontrivial form field, the general solution could be found and is given by eq. (5.22). We have, therefore, constructed a strong coupling version of these generalized rolling radii solutions with a one-form field strength, where the radii now specify the domain wall geometry rather that the size of maximally symmetric subspaces. We stress that the potential U in the time-dependent equations of motion does not originate from the potentials in the action (2.1) but from the nontrivial Ramond-Ramond scalar. The potentials in the action are canceled by the static domain wall part of the solution, as in the previous example.

From the similarity to the known generalized rolling radii solutions, we can also directly infer some of the basic cosmological properties of our solution, using the results of ref. [7, 8]. We expect the integration constants to split into two disjunct sets which lead to solutions in the $(-)$ branch, comoving time range $t \in [-\infty, t_0]$, and the $(+)$ branch, comoving time range $t \in [t_0, \infty]$, respectively. The $(-)$ branch ends in a future curvature singularity and the $(+)$ branch starts in a past curvature singularity. In both branches the solutions behave like rolling radii solutions asymptotically; that is, at $t \to -\infty, t_0$ in the $(-)$ branch and at $t \to t_0, \infty$ in the $(+)$ branch. The two asymptotic regions in both branches have different expansion properties in general and the transition between them can be attributed to the nontrivial form field.

Let us now analyze this in more detail for our solution, following the method presented in ref. [7, 8]. First we should express our solution in terms of the comoving time t by integrating $dt = n(\tau)d\tau$. The gauge parameter $n(\tau)$ is explicitly given by

$$n = e^{(\vec{d}-\vec{q})\cdot\vec{k}}|\tau_1 - \tau|^{-x+\Delta-1}|\tau|^{x-1} \tag{5.31}$$

where

$$x = \vec{d}\cdot\vec{w}\,, \qquad \Delta = \vec{d}\cdot\vec{c}\,. \tag{5.32}$$

Given this expression, the integration cannot easily be performed in general except in the asymptotic regions $\tau \to 0, \tau_1$. These regions will turn out to be precisely the asymptotic rolling-radii limits. Therefore, for our purpose, it suffices to concentrate on those regions. Eq. (5.31) shows that the resulting range for the comoving time depends on the magnitude of Δ and x (note that Δ is a fixed number, for a given model, whereas x depends on the integration constants). It turns out that for all values of the integration constants we have either $x < \Delta$ or $x > 0 > \Delta$. This splits the space of integration constant into two disjunct sets corresponding to the $(-)$ and the $(+)$ branch

as explained before. More precisely, we have the mapping

$$\tau \to t \in \begin{cases} [-\infty, t_0] & \text{for } x < \Delta < 0\,, \quad (-) \text{ branch} \\ [t_0, +\infty] & \text{for } x > 0 > \Delta\,, \quad (+) \text{ branch} \end{cases} \tag{5.33}$$

where t_0 is a finite arbitrary time (which can be different for the two branches). We recall that the range of τ is $0 < \tau < \tau_1$. The above result can be easily read off from the expression (5.31) for the gauge parameter. Performing the integration in the asymptotic region we can express τ in terms of the comoving time and find the Hubble parameters, defined by eq. (4.22), and the powers $\vec{p}$. Generally, we have

$$\vec{p} = \begin{cases} \frac{\vec{w}}{x} & \text{at} \quad \tau \simeq 0 \\ \frac{\vec{w}-\vec{c}}{x-\Delta} & \text{at} \quad \tau \simeq \tau_1 \end{cases} \,. \tag{5.34}$$

Note that, from the mapping (5.33), the expression at $\tau \simeq 0$ describes the evolution powers at $t \to -\infty$ in the $(-)$ branch and at $t \simeq t_0$ in the $(+)$ branch; that is, the evolution powers in the early asymptotic region. Correspondingly, the expression for $\tau \simeq \tau_1$ applies to the late asymptotic regions; that is, to $t \simeq t_0$ in the $(-)$ branch and to $t \to \infty$ in the $(+)$ branch. As before, these powers satisfy the rolling radii constraints (4.24).

Let us now insert the explicit expression for $\vec{d}$, $\vec{w}$ and $\vec{c}$, eqs. (4.15),(5.27) and (5.26), that specify our example into those formulae. First, from eq. (5.32) we find that

$$x = -1 \pm 3\frac{\sqrt{3}}{4}\,, \qquad \Delta = -2\,. \tag{5.35}$$

Note that the space of integration constants just consists of two points in our case, represented by the two signs in the expression for x above. Clearly, from the criterion (5.33) the upper sign leads to a solution in the $(+)$ branch and the lower sign to a solution in the $(-)$ branch. In each branch we therefore have a uniquely determined solution. Using eq. (5.34) we can calculate the asymptotic evolution powers in the $(-)$ branch

$$\vec{p}_{-,t\to-\infty} = \begin{pmatrix} +.06 \\ +.81 \\ +.13 \end{pmatrix}\,, \qquad \vec{p}_{-,t\to t_0} = \begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix}\,. \tag{5.36}$$

Correspondingly, for the $(+)$ branch we have

$$\vec{p}_{+,t\to t_0} = \begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix}\,, \qquad \vec{p}_{+,t\to\infty} = \begin{pmatrix} +.06 \\ +.81 \\ +.13 \end{pmatrix}\,. \tag{5.37}$$

Note that these vectors are in fact the same as in the $(-)$ branch, with the time order being reversed. This happens because they are three conditions on the powers $\vec{p}$ that hold in both

branches, namely the two rolling radii constraints (4.24) and the separation constraint $\beta = \gamma$, eq. (5.7), which implies that $p_3 = 6p_2$. Since two of these conditions are linear and one is quadratic, we expect at most two different solutions for $\vec{p}$. As in the previous solution, the time variation of the Calabi-Yau volume (third entry) is tracking the orbifold variation (second entry) as a consequence of the separation condition and, hence, needs not to be discussed separately. The first entry gives the evolution power for the spatial worldvolume in the five-dimensional Einstein frame. For a conversion to the four-dimensional Einstein frame one should again apply eq. (4.28). It is clear from the above numbers, however, that this conversion does not change the qualitative behaviour of the worldvolume evolution in any of the cases. Having said this, let us first discuss the $(-)$ branch. At $t \to -\infty$ the powers are positive and, hence, the worldvolume and the orbifold are contracting. The solution then undergoes the transition induced by the Ramond-Ramond scalar. Then at $t \simeq t_0$ the worldvolume is still contracting while the orbifold has turned into superinflating expansion. In the $(+)$ branch we start out with a subluminally expanding worldvolume and a contracting orbifold at $t \simeq t_0$. After the transition both subspaces have turned into subluminal expansion.

6 Conclusion

In this paper we have presented the first examples of cosmological solutions in five-dimensional Hořava-Witten theory. They are physically relevant in that they are related to the exact BPS three–brane pair in five dimensions, whose $D = 4$ worldvolume theory exhibits $N = 1$ supersymmetry. A wider class of such cosmological solutions can be obtained and will be presented elsewhere [11]. We expect solutions of this type to provide the fundamental scaffolding for theories of the early universe derived from Hořava-Witten theory, but they are clearly not sufficient as they stand. The most notable deficiency is the fact that they are vacuum solutions, devoid of any matter, radiation or potential stress-energy. Inclusion of such stress-energy is essential to understand the behaviour of early universe cosmology. A study of its effect on the cosmology of Hořava-Witten theory is presently underway [18].

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Extremal Black Hole Entropy

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Introduction

One of the successes of string theory has been an explanation of the entropy of a class of extremal black holes

$$A/4G_N = \ln d_{micro}$$

A: Area of the event horizon

d_{micro}: microscopic degeneracy of the system of branes which carry the same entropy as the black hole.

This formula is quite remarkable since it relates a geometric quantity in space-time to a counting problem.

However the Bekenstein-Hawking formula is an approximate formula that holds in classical general theory of relativity.

– works well only when the charges carried by the black hole are large and hence the curvature at the horizon is small.

The calculation on the microscopic side also simplifies when the charges are large.

Instead of doing exact counting of quantum states, we can use approximate methods which gives the result for large charges.

On the microscopic side we now have a very good understanding of the exact degeneracies of a class of BPS black holes in $N = 4$ and $N = 8$ supersymmetric string theories.

Is there a generalization of the Bekenstein- Hawking formula on the macroscopic side that can be used to calculate the exact black hole degeneracies?

This can then be compared to the exact microscopic results.

For this we shall need to compute two types of corrections:

– higher derivative (α') corrections.

– quantum (string loop) corrections.

Wald's formula gives a method for computing higher derivative interactions to the black hole entropy.

Is there a generalization d_{macro} of this formula in the full quantum theory of gravity that will give the exact degeneracies of black holes?

In general the macroscopic degeneracy, denoted by d_{macro} can have two kinds of contributions:

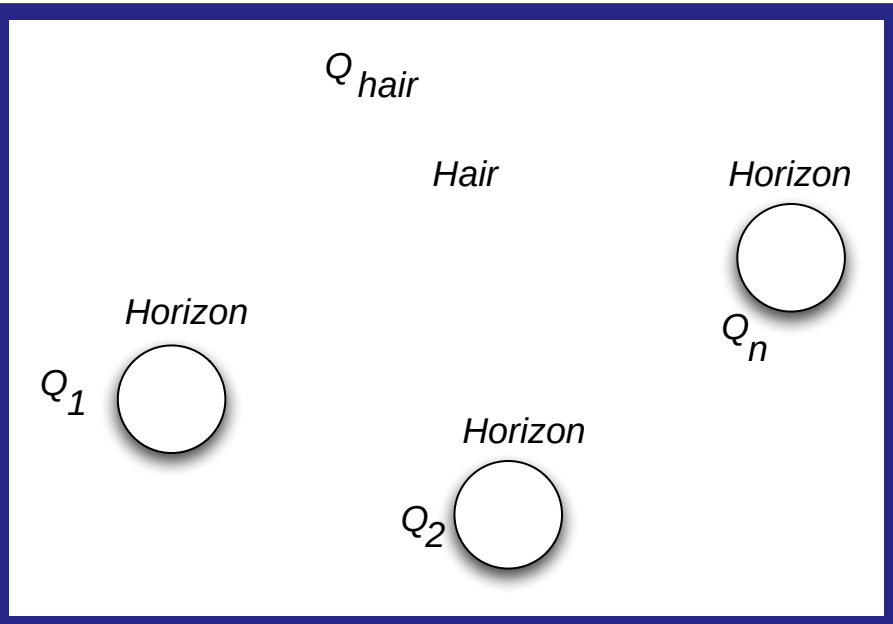
1. From degrees of freedom living outside the horizon (hair)

Example: The fermion zero modes associated with the broken supersymmetry generators.

2. From degrees of freedom living inside the horizon.

We shall denote the degeneracy associated with the horizon degrees of freedom by d_{hor} and those associated with the hair degrees of freedom by d_{hair}.

Our main goal: Find a macroscopic formula for d_{hor}.



The proposed formula for d_{macro}:

$$\sum_{n} \sum_{\substack{\{\vec{Q}_i\},\vec{Q}_{hair} \\ \sum_{i=1}^{n} \vec{Q}_i + \vec{Q}_{hair} = \vec{Q}}} \left\{ \prod_{i=1}^{n} d_{hor}(\vec{Q}_i) \right\} d_{hair}(\vec{Q}_{hair}; \{\vec{Q}_i\})$$

Proposal for d_{hor}

Near horizon geometry of an extremal black hole always has the form of $AdS_2 \times K$.

K: some compact space, possibly fibered over AdS_2.

K includes the compact part of the space time as well as the angular coordinates in the black hole background, *e.g.* S^2 for a four dimensional black hole.

The near horizon geometry is separated from the asymptotic region by an infinite throat and is, by itself, a solution to the equations of motion.

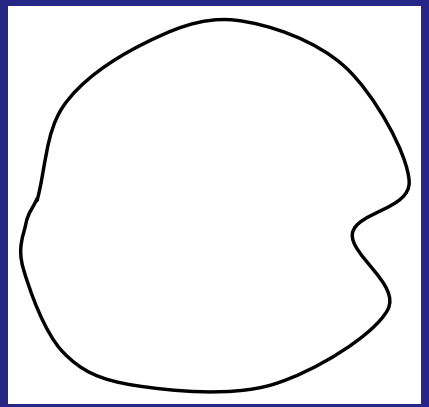
Thus we expect d_{hor} to be given by some computation in the near horizon $AdS_2 \times K$ geometry.

Go to the euclidean formalism and represent the AdS_2 factor by the metric:

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad 1 \leq r < \infty, \quad \theta \equiv \theta + 2\pi$$

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \leq r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

$z = \sqrt{r^2-1}\, e^{i\theta}$ plane:



Proposal for d_{hor} (Quantum entropy function):

$$d_{hor} = Z^{(finite)}$$

$$Z = \left\langle \exp[-iq_k \oint d\theta\, A_\theta^{(k)}] \right\rangle$$

$\langle\ \rangle$: Path integral over string fields in the euclidean near horizon background geometry.

$\{q_k\}$: electric charges carried by the black hole, representing electric flux of the U(1) gauge field $A^{(k)}$ through AdS_2

$\oint$: integration along the boundary of AdS_2

finite: Infrared finite part of the amplitude.

Managing the infrared divergence:

Cut-off: $r \leq r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

$\Rightarrow$ the boundary of AdS_2 has finite length $L \propto r_0$.

$Z^{(finite)}$ is defined by expressing Z as

$$Z = e^{CL+O(L^{-1})} \times Z^{(finite)}$$

C: A constant

Equivalently: $\ln Z^{(finite)} = \lim_{L\to\infty} \left(1 - L\frac{d}{dL}\right) \ln Z$

The definition can be shown to be independent of the choice of $f(\theta)$.

The role of

$$\exp\left[-i\mathbf{q_k}\oint \mathbf{d}\theta\, \mathbf{A}_\theta^{(\mathbf{k})}\right].$$

In computing the path integral over AdS_2 we need to work in a fixed charge sector since the charge mode is non-normalizable and the mode associated with the chemical potential is normalizable.

$\Rightarrow$ We need to add boundary terms in the action to make the path integral consistent.

$\exp\left[-i\mathbf{q_k}\oint \mathbf{d}\theta\, \mathbf{A}_\theta^{(\mathbf{k})}\right]$ **provides the required boundary term.**

Consistency checks:

1. In the classical limit

$$\mathbf{Z} = \exp\left[-\mathbf{A_{bulk}} - \mathbf{A_{boundary}} - i\mathbf{q_k}\oint \mathbf{d}\theta\, \mathbf{A}_\theta^{(\mathbf{k})}\right]$$

evaluated on the attractor geometry.

After extracting the finite part one finds:

$$\mathbf{Z}^{(\mathbf{finite})} = \exp(\mathbf{S_{wald}})$$

$\mathbf{S_{wald}}$**: Wald entropy of the black hole.**

2. By AdS/CFT correspondence $Z = Z_{CFT_1}$.

CFT_1: Quantum mechanics obtained by taking the infrared limit of the brane system describing the black hole.

Since typically this theory has a gap, the infrared limit consists of just the ground states in a fixed charge sector.

$$\Rightarrow Z = d(q)\, e^{-E_0 L}$$

$(E_0, d(q))$: ground state (energy, degeneracy)

Thus $Z^{(finite)} = d(q)$.

Note: $d_{hor} = Z^{(finite)}$ computes the degeneracy for fixed charges, including angular momentum.

Thus this approach always gives the macroscopic entropy in the microcanonical ensemble.

Degeneracy vs index

On the microscopic side we usually compute an index

On the other hand d_{hor} computes degeneracy.

How do we compare the two?

Strategy: Use d_{hor} to compute the index on the macroscopic side.

We shall illustrate this for the helicity trace B_n for a four dimensional single centered black hole.

For a black hole that breaks 2n supercharges we define

$$B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h} (2h)^n$$

h: 3rd component of angular momentum in rest frame

$$B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h_{hor}+2h_{hair}} (2h_{hor} + 2h_{hair})^n$$

In 4D only $h_{hor} = 0$ black holes are supersymmetric

$$\rightarrow B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h_{hair}} (2h_{hair})^n = d_{hor}\, B_{n;hair}$$

If the only hair degrees of freedom are the fermion zero modes associated with the broken suspersymmetry generators then $B_{n;hair} = 1$, and hence $B_n = d_{hor}$.

Comparison with microscopic index

We shall consider quarter BPS dyons in type IIB string theory on K3 $\times$ S^1 $\times$ $\widetilde{S}^1$ and focus on a special class of states containing

D5/D3/D1 branes wrapped on 4/2/0 cycles of K3 $\times$ (S^1 or $\widetilde{S}^1$)

Q: D-brane charges wrapped on 4/2/0 cycles of K3 $\times$ $\widetilde{S}^1$

P: D-brane charges wrapped on 4/2/0 cycles of K3 $\times$ S^1

Q and P are each 24 dimensional vectors.

We shall try to explain some features of the microscopic index of this system using the quantum entropy function.

The relevant index is $B_6(Q, P)$ – the 6th helicity trace index of quarter BPS states carrying charges (Q, P).

Besides depending on the charges, $B_6(Q, P)$ also depends on the asymptotic values of the moduli fields as the degeneracy can jump as we cross walls of marginal stability.

In order to facilitate comparison with the macroscopic results we shall choose the asymptotic moduli such that only single centered black holes contribute to $B_6(Q, P)$.

Duality symmetries

The duality symmetries which take D-branes to D-branes is given by

$$O(4,20;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$$

An arithmetic invariant of $O(4,20;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$:

$$\ell \equiv \gcd\{Q_i P_j - Q_j P_i\}$$

Dabholkar, Gaiotto, Nampuri

With the help of $SL(2,\mathbb{Z})_S$ transformation any charge vector can be brought to the form

$$(Q,P) = (\ell Q_0, P_0), \quad \gcd\{Q_{0i}P_{0j} - P_{0i}Q_{0j}\} = 1$$

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We shall proceed with this choice.

Intersection form of 4/2/0 forms on $K3$ defines additional $O(4,20;\mathbb{Z})$ invariants

$$Q^2, \qquad P^2, \qquad Q \cdot P$$

One finds that for $(Q,P) = (\ell Q_0, P_0)$ the microscopic result for $B_6(Q,P)$ takes the form

$$\sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s), \qquad s|\ell \Leftrightarrow \ell/s \in \mathbb{Z}$$

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$f(m,n,p)$: Fourier transform of the inverse of Igusa cusp form

$$B_6(Q,P) = \sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)$$

Note: For $\ell = 1$ only the $s = 1$ terms contribute.

Dijkgraaf, Verlinde, Verlinde

Our goal will be to try to understand the extra terms for $\ell > 1$ from the macroscopic viewpoint.

For large charges

$$\begin{aligned} & f(Q^2/s^2, P^2, Q\cdot P/s) \\ = \;& \exp\left[\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right] \\ & \times \text{Series expansion in inverse powers of charges} \\ & + \text{Exponentially suppressed corrections} \end{aligned}$$

$$B_6(Q,P) = \sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)$$

Note: $s = 1$ term always contributes.

For large charges this gives the asymptotic expansion

$$\exp\left[\pi\sqrt{Q^2P^2 - (Q\cdot P)^2} + \cdots\right]$$

Macroscopic understanding of $\cdots$ requires loop corrections to the partition function in AdS_2.

– work in progress

Rest of the talk: Understanding the extra terms which appear in the microscopic formula for $\ell > 1$ from the macroscopic viewpoint.

Strategy: Look for additional saddle points in the path integral with the following properties:

1. It must be parametrized by an integer s satisfying the constraint $\ell/\mathbf{s} \in \mathbb{Z}$

2. The classical contribution to $Z^{(\text{finite})}$ from this saddle point must be equal to

$$\exp\left[\pi\sqrt{\mathbf{Q}^2\mathbf{P}^2-(\mathbf{Q}\cdot\mathbf{P})^2}/\mathbf{s}\right]$$

3. It must preserve sufficient amount of supersymmetry so as not to vanish due to fermion zero mode integration.

There are indeed such saddle points in the path integral, constructed as follows.

1. Take the original near horizon geometry of the black hole.

2. Take a $\mathbb{Z}_S$ orbifold of this background with $\mathbb{Z}_S$ acting as

a) $2\pi/s$ rotation in AdS_2

a) $2\pi/s$ rotation in S^2

c) $2\pi/s$ unit of translation along the circle S^1.

– freely acting $\mathbb{Z}_s$.

Banerjee, Jatkar, A.S.; A.S.; Murthy, Pioline

1. Quantization of the RR 3-form flux requires that $\ell/s \in \mathbb{Z}$

2. Ordinarily an orbifold of this type will change the asymptotic structure of space time but in AdS_2 it preserves the asymptotic boundary conditions.

3. The contribution to Z from this saddle point is given by

$$e^{CL} \exp\left[\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s\right]$$

Thus its contribution to $Z^{(finite)}$ is of order

$$\exp\left[\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s\right]$$

Furthermore these saddle points preserve sufficient amount of supersymmeries so that integration over the fermion zero modes associated with the broken supersymmetries do not make the path integral vanish automatically.

Banerjee, Banerjee, Gupta, Mandal, A.S.

Orbifold action:

$$\theta \to \theta + 2\pi/s, \quad \phi \to \phi + 2\pi/s, \quad x^5 \to x^5 + 2\pi/s$$

At AdS_2 center ($r = 1$) the shift in θ is irrelevant.

$\to$ the identification is $(\phi, x^5) \equiv (\phi + 2\pi/s, x^5 + 2\pi/s)$.

Thus the RR flux Q through the cycle at $r = 1$, spanned by (x^5, ψ, ϕ) gets divided by s.

Flux quantization $\to$ the orbifold is well defined only if Q is divisible by s, i.e. if

$$\ell/s \in \mathbb{Z}$$

Denoting the (r, θ, ϕ, x^5) coordinates of the orbifold by $(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{x}^5)$ we get the new metric

$$ds^2 = v\left[\frac{d\tilde{r}^2}{\tilde{r}^2 - 1} + (\tilde{r}^2 - 1)d\tilde{\theta}^2\right] + w\left[d\psi^2 + \sin^2\psi d\tilde{\phi}^2\right] + \cdots$$

$$(\tilde{\theta} + 2\pi/s, \tilde{\phi} + 2\pi/s, \tilde{x}^5 + 2\pi/s) \equiv (\tilde{\theta}, \tilde{\phi}, \tilde{x}^5)$$

Define

$$\theta = s\tilde{\theta}, \quad r = \tilde{r}/s, \quad \phi = \tilde{\phi} - \tilde{\theta}, \quad x^5 = \tilde{x}^5 - \tilde{\theta}$$

Then

$$ds^2 = v\left(\frac{dr^2}{r^2 - s^{-2}} + (r^2 - s^{-2})\, d\theta^2\right) + w[d\psi^2 + \sin^2\psi(d\phi + s^{-1}d\theta)^2]$$

$$(\theta + 2\pi, \phi, x^5) \equiv (\theta, \phi, x^5)$$

Its contribution to $d_{hor}(Q, P)$ in the classical limit is given by

$$\exp[S_{wald}/s] = \exp\left[2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right]$$

This is the same behaviour as of $f(Q^2/s^2, P^2, Q\cdot P/s)$.

Note: The infrared divergent part is also divided by s and gives a contribution to the exponent:

$$C\widetilde{L}/s = CL + O(L^{-1})$$

Final comments

In principle one should be able to reproduce the full macroscopic partition function from path integral over the near horizon $AdS_2 \times K$ geometry.

This would seem to be difficult task as it involves path integral over all the string fields.

However one can argue that supersymmetry restricts the path integral to over configurations preserving a certain amount of supersymmetry.

Beasley, Gaiotto, Guica, Huang, Strominger, Yin

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Hope: Using this result one can collapse the whole path integral to a finite dimensional integral which can then be computed.

In this context it is amusing to note that even the microscopic degeneracy formula in this theory can be expressed as a sum of contributions over different saddle points, with the contribution from each saddle point being given by a two dimensional integral.

Once we are confident that the formalism works for N=4 black holes, we can then use it to compute the degeneracies of N=2 black holes where the microscopic formula is still unknown.

Reissner-Nordstrom solution in $D = 4$:

$$\begin{aligned}\mathbf{ds^2} &= -(\mathbf{1}-\rho_+/\rho)(\mathbf{1}-\rho_-/\rho)\mathbf{d}\tau^2 \\ &\quad + \frac{\mathbf{d}\rho^2}{(\mathbf{1}-\rho_+/\rho)(\mathbf{1}-\rho_-/\rho)} \\ &\quad + \rho^2(\mathbf{d}\theta^2 + \sin^2\theta \mathbf{d}\phi^2)\end{aligned}$$

Define

$$\mathbf{2}\lambda = \rho_+ - \rho_-, \quad \mathbf{t} = \frac{\lambda\tau}{\rho_+^2}, \quad \mathbf{r} = \frac{\mathbf{2}\rho - \rho_+ - \rho_-}{\mathbf{2}\lambda}$$

and take $\lambda \to 0$ limit.

$$\mathbf{ds^2} = \rho_+^2\left[-(\mathbf{r^2}-\mathbf{1})\mathbf{dt^2} + \frac{\mathbf{dr^2}}{\mathbf{r^2}-\mathbf{1}}\right] + \rho_+^2(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\phi^2)$$

On extremal black holes

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We take a fresh look at the viability of physically realistic extremal black holes within our (non-supersymmetric) low energy physics. By incorporating prefactors and volume effects, we show that Schwinger discharge in charge neutral environments is far more efficient than commonly assumed. Using ionization estimates for neutral hydrogen, we obtain a new and robust lower bound on the mass of an extremal electrically charged black hole, exceeding $10^{14} M_\odot$. For magnetic black holes, we compute the Lee-Nair-Weinberg instability and revisit early universe pair creation rates, including singular instantons that substantially enhance production, to demonstrate that the extreme charges required for stability are cosmologically implausible. Finally, we suggest that an extremal Kerr black hole could shed angular momentum via superradiant scattering from the stochastic gravitational wave background. Taken together, our results provide a unified picture that extremal black holes of any type are unlikely to persist in our universe.

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I. INTRODUCTION

Extremal black holes remain among the most perplexing solutions in classical general relativity. They often sit at the boundary between naked singularities and well-behaved spacetimes, challenge our understanding of black hole thermodynamics, and yet we do not even know if they exist beyond our mathematical expressions. In these objects, inner and outer horizons coincide, and they reach the maximum charge/angular momentum allowed for a black hole. Their vanishing Hawking temperature suggests a connection to the ill-understood third law of black hole thermodynamics, whereas their finite area raises puzzles with the conventional interpretation of entropy as a counting of microstates. It is precisely their unique thermodynamic character, as well as their threshold nature in terms of cosmic censorship that makes them so compelling a study.

In a recent essay, Dafermos [1] reflects on these aspects of extremality, both discussing the historical development of our understanding as well as how and why that understanding was incomplete. The third law of black hole thermodynamics, as conjectured in [2], simply asserted that it should be impossible to reduce the surface gravity of a black hole to zero in a finite sequence of operations – such a law would not only keep the thermodynamical analogy for black holes intact, but also would ensure that one could not, in a finite set of steps, push a black hole beyond extremality, rendering it a naked singularity. In 1986, Israel [3], provided a definitive formulation of this third law, then presented a proof that there was no *continuous* process in which the black hole could be rendered extremal while maintaining the weak energy condition. However, the results of Sorce and Wald [4], while focused on proving that no gedanken experiment can succeed in overcharging or overspinning a Kerr-Newman black hole,suggest that an extremal black hole can be reached from a near-extremal one via carefully tuned perturbations, assuming only the null energy condition. Indeed recently, Kehle and Unger [5] have shown how to form an extremal Reissner-Nordström black hole through gravitational collapse in Einstein-Maxwell-charged scalar field theory – we refer the reader to [1] for a comprehensive yet concise critical evaluation of these results and how Israel's original proof is evaded.

Beyond their thermodynamic peculiarities, extremal black holes play a key role in fundamental high-energy physics. They often emerge as supersymmetric gravitational solutions in string theory [6, 7]. Indeed, a recent tantalising result of Brown et al. [8] explores the partition function of near extremal black holes and, via an exact computation in JT gravity, argue that as one approaches extremality, quantum gravitational effects strongly modify the partition function, suppressing the entropy. This would appear to resolve the puzzle of an extremal black hole having finite area yet being at zero temperature that was debated some years ago in the literature (see e.g. [9, 10]). The notion that an extremal, or near extremal, black hole should receive strong quantum corrections then raises the question of whether these modifications could be observable, and indeed could provide a clear signal of these corrections [11, 12] – a tantalising prospect indeed!

The discussion of Dafermos (and indeed many explorations of over-charging or spinning black holes such as [4, 13, 14]) all treat the black hole in splendid isolation – the spacetime is determined by classical general relativity with test particles or Maxwell fields. In our universe however, black holes are not isolated systems, therefore we critically re-examine the question: can extremal black holes exist in our universe? This is a crucial question, as the answer determines whether such geometries can serve as a potential testing ground for quantum gravity and black hole thermodynamics, or whether extremality is inherently unreachable, with black holes naturally evolving away from it, rendering these spacetimes inaccessible and ultimately unobservable.

The possibility of long-lived electrically charged black holes is often dismissed on the grounds that the Coulomb force vastly exceeds gravity, implying that charged particles in any surrounding plasma would rapidly neutralize the black hole [15]. Here we consider the viability of physically realistic charged extremal black holes immersed in a *neutral* environment (the QED vacuum and neutral hydrogen). We first revisit the Gibbons [16] Schwinger discharge process, re-evaluating the probability of discharge including 1-loop effects and a proper consideration of the probability of discharge. We then derive an even stronger constraint from the ionization of matter in the local environment. In both cases, the impact of the prefactor and volume in which electric discharge occurs is extremely significant. We then evaluate the likelihood of extremal magnetic black holes, that might be expected to be stable to any pair production decay, computing the details of the linear instability first argued by Lee, Nair and Weinberg [17]. We also revisit arguments on black hole creation during inflation, noting a broader range of instantons than previously considered. We conclude with remarks on Kerr extremality.

A remark on units

Throughout our discussion, we will, in the main, use natural units as is standard convention in GR literature, i.e. $\hbar = c = 4\pi\epsilon_0 = 1$. We will also use the standard (as opposed to reduced) definitions of Planck units, setting $G = 1$. However when discussing the details of physical processes, where scales and numerical estimates are relevant, we restore these constants in the presentation of results, using SI units.

II. ELECTRICALLY CHARGED EXTREMAL BLACK HOLES

We first discuss the feasibility of an extremal electrically charged black hole. A classic result by Gibbons [16] calculated how the electrostatic field of an electrically charged black hole can induce Schwinger pair production, leading to charge depletion. The essence of this effect is that the energy of an electric field is proportional to $|\mathbf{E}|^2$, thus the stronger the electric field, the greater the drop in energy if the field discharges an amount $\delta\mathbf{E}$. Meanwhile, to create an electron/positron pair, we require (at minimum) a rest mass energy of $2m_e c^2$. An estimate of the drop in the electric field over a distance d due to the pair would have the dependence $\sim |\mathbf{E}_0| ed$, where we can guesstimate d using the uncertainty relation $\hbar \sim \delta E \delta t \sim m_e c^2 d/c$. Thus we expect pair creation to start to discharge the electric field at a magnitude around

$$|\mathbf{E}_0| \sim \mathcal{O}\left(\frac{m_e^2 c^3}{e\hbar}\right) \tag{1}$$

consistent with the Schwinger result for the decay rate per unit volume:

$$\Gamma_{\text{Schwinger}} = \frac{(e|\mathbf{E}_0|)^2}{4\pi^3\hbar^2 c}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left[-\frac{n\pi m_e^2 c^3}{\hbar e|\mathbf{E}_0|}\right] \tag{2}$$

Putting this into the context of a charged black hole, the electric field scales approximately as Q/r_h^2, the charge of the black hole divided by the square of the horizon radius. For an extremal black hole, $r_h \sim Q \sim M$, hence $|\mathbf{E}| \propto 1/M$. Thus, smaller black holes will have stronger local electric fields and will be more likely to discharge leading to a lower bound on the allowed mass of an extremal black hole. This is the intuition behind the argument of Gibbons, however the results / estimates above were all obtained without consideration of gravity, or the geometry of the black hole – indeed the original motivation of Gibbons was to understand particle production processes in strong gravitational fields.

In order to determine the discharge of an electrically charged black hole, Gibbons followed a similar argument to that of Hawking [18] – computing the Bogoliubov coefficients between in and out states modelled by with a charged scalar quantum field for simplicity. By considering small and large black hole mass limits, he confirmed the validity of the Schwinger argument for large black hole masses (as might be anticipated from the small local spacetime curvature), and confirmed that small black holes would indeed discharge rapidly.

We therefore revisit the Schwinger discussion for large extremal black holes, noting the importance of the prefactor, which has been overlooked in the literature for this purpose to the best of our knowledge. Recall that the Schwinger 1-loop computation gives a probability per unit volume per unit time. In order to decide how probable it is therefore for a given extremal black hole to discharge one unit of charge, we must multiply by the volume in which the discharge can take place, and a representative time. The volume will clearly be given by the area of the black hole multiplied by a radial factor over which the decay rate remains approximately similar, and for simplicity we will take the representative time as the light-crossing time for the black hole, r_h/c. Note that this time is very small on astrophysical scales, making the bound that we get very conservative.

While we might expect r_h to be representative of the extent of the radial range over which the Schwinger amplitude is similar, we also note a recent study of the radial dependence of the Schwinger amplitude for an extremal black hole

by Lin and Shiu [19], who used a worldline instanton approach to carefully follow the exponent and prefactor of pair production outside the horizon (though these were explored for particles whose charge:mass ratio was of order unity, rather than the 10^{22} factor of the electron). We can extract the general properties of the radial dependence from their work. From Figure 1 in [19], one sees that the exponent roughly doubles at a distance of half the horizon radius from the horizon, which is broadly in keeping with the behaviour of the electric field in the exponent. Similarly, in Figure 4 they present the radial behaviour of the prefactor as one increases the charge:mass ratio, from which we see that the prefactor increases with increasing charge:mass ratio and drops inversely as one moves away from the horizon. These properties are consistent with the exponent in the Schwinger result where one substitutes the behaviour of the electric field of the extremal black hole.

We therefore evaluate the Schwinger amplitude at the event horizon of the black hole, and multiply by r_h^4/c to get an overall estimate of the likelihood of an extremal black hole to discharge. One might be concerned at using an expression for volume in terms of the $\{t, r\}$ coordinates when the extremal RN solution has an infinite "throat". Strictly speaking, one should perform a volume integral across a segment of the future event horizon extending over a range of advanced time and extending away from the horizon. The local Kruskals are distinct from the usual expressions [20], and were derived by Carter [21], but it is straightforward to transform to this system and confirm that r_h^4/c captures the correct behaviour of this volume.

Restoring units, the location of the horizon of an RN black hole is given by

$$1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi\varepsilon_0 c^4 r^2} = 0 \,, \tag{3}$$

thus the extremal limit occurs when

$$Q = \sqrt{4\pi\varepsilon_0 G} M \quad ; \qquad r_h = \frac{GM}{c^2} = \sqrt{\frac{G}{4\pi\varepsilon_0}}\frac{Q}{c^2} \tag{4}$$

Meanwhile, the electric field for a black hole with charge Q is

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{5}$$

Thus, substituting in (2) we obtain:

$$\Gamma_{\text{Schwinger}} = \frac{(eQ)^2}{64\pi^5\varepsilon_0^2\hbar^2 c r^4}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left[-\frac{4n\pi^2\varepsilon_0 m_e^2 c^3 r^2}{\hbar e Q}\right] \tag{6}$$

Thus, multiplying by the volume factor, the overall likelihood of the black hole discharging (focussing on the near horizon region) is:

$$\begin{aligned}\mathcal{P} = \frac{r_h^4}{c}\,\Gamma_{\text{Schwinger}} &= \frac{(eQ)^2}{64\pi^5\varepsilon_0^2\hbar^2 c^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left[-\frac{n\pi m_e^2 G Q}{\hbar e c}\right] \\ &= \frac{\alpha^2}{4\pi^3}\frac{Q^2}{e^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp\left[-\frac{4n\pi^3 L_p^2}{\lambda_e^2}\frac{Q}{e}\right]\end{aligned} \tag{7}$$

where $\alpha = e^2/4\pi\varepsilon_0\hbar c$ is the fine structure constant, $L_p^2 = G\hbar/c^3$ is the Planck length, and $\lambda_e = 2\pi\hbar/m_e c$ is the Compton wavelength of the electron.

Expressed in this form, the hierarchy between the Planck length and the Compton wavelength of the electron indicates the large strength of the electric field required to damp the discharge (recall this is an extremal black hole so that the strength of the electric field varies inversely to the black hole size) but also note the prefactor. The value of the fine structure constant gives a small numerical factor, $\alpha^2/4\pi^3 \sim 4\times 10^{-7}$, however, Q/e is extremely large before the exponential factor damps the pair creation. The critical electric field is therefore set by the Lambert function, or the inverse of ze^z, for the primary $n = 1$ term in the Schwinger expression: $\mathcal{P} \approx 1$ for

$$\begin{aligned}\left(-2\pi^3\frac{L_p^2}{\lambda_e^2}\frac{Q}{e}\right)\exp\left[-2\pi^3\frac{L_p^2}{\lambda_e^2}\frac{Q}{e}\right] &= -\frac{4\pi^4\sqrt{\pi}L_p^2}{\alpha\lambda_e^2} \\ \Rightarrow \qquad -2\pi^3\frac{L_p^2}{\lambda_e^2}\frac{Q}{e} &= W_{-1}\left(-\frac{4\pi^4\sqrt{\pi}L_p^2}{\alpha\lambda_e^2}\right) \\ \Rightarrow \qquad \frac{Q}{e} &\sim 3.6\times 10^{46}\end{aligned} \tag{8}$$

This value corresponds to $M \sim 3.4 \times 10^7 M_\odot$, or a couple of orders of magnitude higher than the threshold value obtained by considering the exponent dependence alone. Thus, only the most massive supermassive black holes are sufficiently large to evade the Schwinger process.

At the time of Gibbons' paper (1975), the existence of black holes was under debate, at least within the gravity theory community – witness the famous wager between Stephen Hawking and Kip Thorne in that same year as to whether black holes existed in our universe! However, in general it was thought that black holes would mostly form from gravitational collapse of stars at the end of their lifetimes, so such a large limit for the discharge of a black hole effectively meant that extremal black holes were just not a possibility. Indeed, in [22], Carr and Hawking, while allowing for the possibility of *primordial black holes* formed in the very early universe, also argued that black holes would not accrete substantially, hence very large mass black holes were unlikely.

The modern perspective however is very different. Although extremely massive black holes had been posited to sit at the centre of galaxies by astronomers for some years (see e.g. [23]), the community at large accepted this as fact only after the observation of stars, in particular S2, orbiting the centre of the Milky Way provided the irrefutable evidence that clinched the case for *supermassive black holes* (SMBH) [24, 25] (though it took Hawking another 6 years to concede his bet with Thorne). Since then, the observational evidence for black holes has become overwhelming, with the LIGO gravitational wave data from black hole mergers [26] and the Event Horizon Telescope imaging of the central SMBHs in our own galaxy and M87 [27, 28] – as well as exquisite imaging of the stellar motion around our own black hole by the Gravity consortium [29].

Since such large mass black holes are now believed to be commonplace, it is (very) theoretically possible that there could be an extremal black hole that might exceed the Schwinger bound and remain extremal, or close to (see the discussion above and also [8]). However we now argue that the required electric field for protection from discharging is in fact even *lower* than the Schwinger bound, as black holes in our universe are not in vacuum, but in an environment with dust and gas. Indeed, the very observations that confirm their existence rely on their interactions with their surroundings: their gravitational influence shapes the orbits of nearby stars, while their accretion disks – composed primarily of hydrogen and helium, along with heavier elements in trace amounts – emit radiation across the electromagnetic spectrum. Imaging of SMBHs captures photons emitted by this infalling material as it heats up and radiates. Therefore, even if the Schwinger pair production is inefficient, discharge can still proceed via ionization of the surrounding matter. Thus, although the electric field of an extremal SMBH is weaker than that of a smaller one, it remains sufficiently strong to ionize hydrogen − the most abundant element in galactic nuclei and accretion environments.

For simplicity, we assume the local galactic matter to be Hydrogen, and will consider neutral Hydrogen as one might expect ionized gas to accrete differentially thus discharging the black hole [15].

The tunnelling ionization rate, p, for an individual hydrogen atom in its ground state in a strong electric field $\mathcal{E}$ can be read off from [30] as:

$$\begin{aligned} p &= \frac{4m_e^3 e^9}{(4\pi\epsilon_0)^5 \hbar^7 \mathcal{E}} \exp\left(-\frac{2m_e^2 e^5}{3(4\pi\epsilon_0)^3 \hbar^4 \mathcal{E}}\right) \\ &= 32\pi^3 \alpha^5 \frac{cL_p^2}{\lambda_e^3}\frac{Q}{e} \exp\left(-\frac{8\pi^2\alpha^3}{3}\frac{L_p^2}{\lambda_e^2}\frac{Q}{e}\right) \end{aligned} \tag{9}$$

Note the similarity to the Schwinger amplitude (7), but this exponent has the additional damping factor of α^3.

As with the pair creation estimate, note that this is the probability of ionization of a given atom per unit time. Therefore, as before, to properly estimate the likelihood of black hole discharge, we take into account the number of atoms in the vicinity of the black hole, and a representative timescale. The number density of hydrogen atoms is around $n \sim 10^6 cm^{-3}$ [31], and for a representative volume we take r_h^4/c as before. Thus, we must multiply the individual rate of ionization by the factor

$$n\frac{r_h^4}{c} = n\frac{G^2Q^4}{(4\pi\varepsilon_0)^2 c^5} = n\alpha^2 \frac{L_p^4}{c}\left(\frac{Q}{e}\right)^4 = \mathcal{N}\frac{\alpha^2 L_p}{c}\left(\frac{Q}{e}\right)^4 \tag{10}$$

where $\mathcal{N} = nL_p^3 \sim 4 \times 10^{-93}$ is the number of atoms per Planck volume near the black hole.

Putting this together, we see the black hole will discharge unless

$$
\begin{aligned}
& np\frac{r_h^4}{c} = 32\pi^3\alpha^7\mathcal{N}\frac{L_p^3}{\lambda_e^3}\left(\frac{Q}{e}\right)^5 \exp\left[-\frac{8\pi^2\alpha^3}{3}\frac{L_p^2}{\lambda_e^2}\frac{Q}{e}\right] < 1 \\
\text{i.e.} \qquad & 2\pi^{3/5}\alpha^{7/5}\mathcal{N}^{1/5}\frac{L_p^{3/5}}{\lambda_e^{3/5}}\frac{Q}{e}\exp\left[-\frac{8\pi^2\alpha^3}{15}\frac{L_p^2}{\lambda_e^2}\frac{Q}{e}\right] < 1 \\
\Rightarrow \qquad & \frac{Q}{e} = -\frac{15\lambda_e^2}{8\pi^2\alpha^3 L_p^2}W_{-1}\left(-\frac{4}{15}\sqrt[5]{\frac{\pi^7\alpha^8 L_p^7}{\mathcal{N}\lambda_e^7}}\right) = 4.77\times 10^{53}
\end{aligned}
\tag{11}
$$

which gives a lower bound of $M \sim 4.5\times 10^{14} M_\odot$ for avoiding discharge. Given that the most massive SMBH (reported recently in [32]) is only 36 billion solar masses, this bound of nearly half a quadrillion solar masses, while comfortably below the total matter in the universe, is nonetheless well beyond current conceptual limits for a massive SMBH.

III. MAGNETIC CHARGE AND EXTREMALITY

Since the Reissner-Nordström solution (from a purely geometric point of view) can be sourced by the energy momentum tensor for both electric and magnetic charges, we now consider whether an extremal magnetic black hole is a realistic prospect in our universe. Thus far, we have no direct evidence of a magnetic 'pole', however, the topology of symmetry breaking indicates that monopoles are generically formed. The existence of a monopole is predicated on a nontrivial Π_2 of the vacuum, which in turn is related to a residual unbroken $U(1)$, that we have in electromagnetism. This means that monopoles are expected to be a by-product of phase transitions in the early universe for generic grand-unified symmetry breaking [33] – indeed, the production of monopoles in the early universe was considered so problematic it was one of the motivations behind the inflationary paradigm. Since the scale of the mass and charge of the monopole are set by the grand unified scale, we expect that discharge by pair creation will be enormously suppressed compared to the electric case, and indeed the computed rate [34] becomes significant only for black holes that are already unstable to a process we now discuss.

The monopole is a nontrivial field configuration in which the field at infinity wraps around the non-contractable 2-cycles in the vacuum manifold. While the details of this field configuration will depend on the details of the gauge group, the essence of the monopole is captured by the solution of t' Hooft and Polyakov [35, 36], where the simplest manifestation of the solution is constructed. In this model with a scalar and non-abelian vector field, the scalar is in the adjoint of SO(3), so that there is a very direct and intuitive picture of the sphere at infinity wrapping the internal sphere of the vacuum. The Lagrangian is:

$$
\mathcal{L}_{\rm m} = \frac{1}{2}D_\mu\mathbf{\Phi}\cdot D^\mu\mathbf{\Phi} - V(|\mathbf{\Phi}|) - \frac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu},
\tag{12}
$$

where

$$
\begin{aligned}
D_\mu\mathbf{\Phi} &= \partial_\mu\mathbf{\Phi} - g\mathbf{A}_\mu\times\mathbf{\Phi} \\
\mathbf{F}_{\mu\nu} &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - g\mathbf{A}_\mu\times\mathbf{A}_\nu,
\end{aligned}
\tag{13}
$$

Here, boldface vector notation refers to the internal SU(2) structure, "$\cdot$" refers to the standard inner product on $\mathbb{R}^3$, and the potential $V(|\mathbf{\Phi}|) = \lambda(\mathbf{\Phi}\cdot\mathbf{\Phi} - \eta^2)^2/4$ has a minimum at $|\mathbf{\Phi}| = \eta$, with η the symmetry breaking scale. The theory contains nonsingular monopoles with magnetic charge P (considered to be in $[A\,m]$ units when restoring the natural constants) and mass $M_{\rm mon}$ given by:

$$
P = \frac{N}{g}, \qquad M_{\rm mon} \sim \frac{4\pi\eta}{g},
\tag{14}
$$

where N is the winding number of the map into the vacuum manifold at infinity. For a magnetic charge with winding number N, the matter fields, up to a gauge transformation, have the asymptotic form

$$
\mathbf{\Phi}_\infty = \eta\,\hat{\mathbf{e}}(\theta,\phi)\,, \qquad \mathbf{A}_{\infty\mu} = (1/g)\,\hat{\mathbf{e}}\times\partial_\mu\hat{\mathbf{e}}
\tag{15}
$$

where $\hat{\mathbf{e}} = (\sin\theta\cos N\phi, \sin\theta\sin N\phi, \cos\theta)$. Consequently, the field strength tensor takes the form $\mathbf{F}_{\theta\phi} = -\mathbf{F}_{\phi\theta} = (N/g)\sin\theta\,\hat{\mathbf{e}}$, which lies entirely in the electromagnetic U(1) subgroup defined by the scalar field and precisely reproduces the radial magnetic field expected in the Maxwell–Einstein theory.

The monopole solution, for $N = 1$ at least, is obtained by assuming a radial profile for the magnitude of the scalar and vector fields [37]:

$$\mathbf{\Phi} = \frac{H(r)}{gr}\hat{\mathbf{e}}(\theta,\phi) \quad , \qquad \mathbf{A}_\mu = \frac{(1-K(r))}{gr}\,\hat{\mathbf{e}}\times\partial_\mu\hat{\mathbf{e}} \tag{16}$$

where generically H and K are found numerically, although in the $\lambda \to 0$, or BPS [37, 38] limit, they have an exact analytic form:

$$H = \frac{r}{r_m}\coth(r/r_m) - 1 \quad ; \qquad K = \frac{r}{r_m \sinh(r/r_m)} \tag{17}$$

where r_m is a lengthscale representing the effective size of the "core" of the monopole, where the SU(2) symmetry is restored. At finite λ, the $1/r$ fall-off in the non-trivial profiles of $\mathbf{\Phi}$ and $\mathbf{A}_\mu$ becomes exponential, with the width set by $1/\sqrt{\lambda}\eta$ and $1/g\eta$ respectively (for winding number 1). The monopole therefore has a dense scalar and (nonabelian) vector core with the mass depending primarily on g and weakly on λ/g^2 [39].

Turning to the magnetically charged black hole, if the black hole radius is larger than this effective core size, the exterior (vacuum) monopole solution can be superimposed on the geometry, however, as the black hole mass is dialled down (or the monopole charge cranked up) something interesting happens: it is now possible to have part of the monopole core peeping out from the horizon – in effect to have the black hole residing within the monopole [40], with the SU(2) fields partially screening the magnetic charge.

Since both magnetically charged Reissner-Nordström and the dressed black hole are possible solutions, Lee, Nair and Weinberg tested the stability of the Reissner-Nordström solution in the SU(2) theory [17]. By considering fluctuations in the scalar and vector fields outside the horizon, they showed that the perturbation equations could be manipulated into a Schrödinger form, allowing them to prove the existence of an unstable mode via a test-function estimation, although they did not compute the actual wave function or timescale. Since we are interested here in timescales, we briefly outline their argument before computing the detail of the instability.

First, let us derive an estimate of where the instability might set in for a simple ($N = 1$) monopole. We might expect this to happen when the horizon radius drops below the size of the monopole core:

$$r_h = M + \sqrt{M^2 - P^2} \lesssim r_m \approx \frac{1}{g\eta} \tag{18}$$

however, assuming the charge is entirely magnetic and due to the monopole, we have $P = 1/g$, hence we deduce that an instability can only emerge if $1/g \leq M < \mathcal{O}(1/g\eta)$. Rearranging (18) we therefore see that an estimate of the emergence of an instability will be when the black hole has a mass approximately below

$$M \lesssim \frac{1}{2g\eta} + \frac{\eta}{2g} \tag{19}$$

To test this intuition, Lee, Nair, and Weinberg, motivated by the Prasad-Sommerfeld Ansatz [37], considered spherically symmetric perturbations around the monopole vacuum (15) of the form (16) but with H and K now functions of r and t. They found that the instability resided in the vector component:

$$K(r,t) \sim e^{\Omega t}u(r) \tag{20}$$

with an equation of motion

$$-\Omega^2 u = -\frac{d^2u}{dr_\star^2} + U(r_\star)u \tag{21}$$

where $r_\star = \int dr/f(r)$ is the tortoise coordinate ($f(r) = 1 - 2M/r + P^2/r^2$ being the Reissner Nordström potential) and

$$U = -\left(1 - \frac{2M}{r} + \frac{P^2}{r^2}\right)\frac{(1 - g^2\eta^2r^2)}{r^2} \tag{22}$$

with $r = r(r_\star)$ implicitly understood.

With this perturbation equation in Schrödinger form, it is easy to see the instability corresponds to a bound state solution, which Lee et al. argued the existence of using trial eigenfunctions. Since $\eta \sim 10^{-4}$ for a typical grand unified theory (GUT) breaking scale, we see that U is negative for $r_h \lesssim 1/g\eta$ in agreement with the heuristic argument above.

When $N > 1$, the estimate above, (19), for the onset of the instability becomes

$$M \lesssim \frac{\sqrt{N}}{2\eta g} + \frac{2\pi\eta N\sqrt{N}}{g} \tag{23}$$

which also implies a bound of the winding number of $N \lesssim 1/\eta^2$. In detail, the potential in the instability equation is modified by replacing the $g^2\eta^2 r^2$ term in brackets with $Ng^2\eta^2 r^2$ [17]. Strictly, (21) only picks up an unstable s-mode, however Lee et al. put forward a heuristic picture in which, as the fields condense they localize at points on the horizon, and they conjectured that these regions could grow into distinct lumps and detach as individual magnetic monopoles, thus reducing the total magnetic charge (a related process was considered by Maldacena [41]). Eventually, when only a single unit of charge remains, the horizon may shrink further by Hawking evaporation, exposing the monopole core.

To see how the instability depends on the physical parameters, consider the rescaled equation

$$\frac{1}{\hat{f}}\frac{d}{d\hat{r}}\frac{1}{\hat{f}}\frac{d}{d\hat{r}}\,u(\hat{r}) = \left[\hat{\Omega}^2 + \frac{(1-\hat{r}^2)}{\hat{r}^2}\right] u(\hat{r}) = \hat{U}(\hat{r})\,u(\hat{r}) \tag{24}$$

where

$$\hat{r} = g\eta r \;\;,\quad \hat{\Omega} = \Omega/g\eta \;\;,\quad \hat{M} = g\eta M \;\;,\quad \hat{f} = \left(1 - \frac{2\hat{M}}{\hat{r}} + \frac{N^2\eta^2}{\hat{r}^2}\right) \tag{25}$$

Clearly, for $\eta \sim 10^{-4}$, the lower bound on $\hat{M}$, $N\eta$, is small, and expanding in $\hat{M}$ shows that the minimum of $\hat{U}$ occurs at $r_{\min} \in [2\hat{M}, 3\hat{M})$, (with the lower bound of $r_{\min}$ occurring at the extremal limit, and the upper at close to zero charge), with a minimum value of

$$\hat{U}_{\min} \in \left[-\frac{1}{16\hat{M}^2}, -\frac{1}{27\hat{M}^2}\right) \tag{26}$$

(again with the lower bound corresponding to the extremal limit). Trial eigenfunctions show that the eigenvalue corresponding to the instability is well approximated by $\hat{\Omega} = a\sqrt{|\hat{U}_{\min}|}$, where a is very close to 1. Unravelling the rescaling, we therefore expect that the timescale of the instability $\Omega \propto 1/M$, and independent of N away from the extremal limit. At, or close to, the extremal limit, since M is bounded below by the charge, we expect that Ω will be roughly inversely proportional to N through the dependence of M. We therefore do not need to perform a large exploration of parameter space, as the behaviour of the rescaled parameters informs us of how the eigenvalues depend on the black hole parameters.

We solved the wave equation for u using a shooting method, expanding the solution at both boundaries in terms of Ω, integrating in to a matching intermediate radius, where demanding that the wave function is C_1 determines Ω. Figure 1 shows the instability frequencies obtained by solving the perturbation equation (21) for a range of black hole masses and winding numbers N.

The plot shows clearly that at fixed mass, the higher charged black hole is more unstable, as we might expect. Clearly, at fixed charge, the extremal black hole is the most unstable, as expected, as this is the smallest horizon size. As the mass increases, the black hole becomes less unstable in line with expectations: once the condition in Eq. (23) is no longer satisfied, the black hole becomes stable and Ω drops to zero. Extremal black holes also become more stable as the charge increases, until the maximum charge is attained. Since extremal black holes are not expected to evaporate, having zero temperature, we see that above this critical threshold, the extremal magnetic black hole is stable.

Note that the instability condition on M, (23), for the extremal black hole can be rewritten as

$$\frac{2\sqrt{N}}{g\eta}\left[\left(x - \frac{1}{4}\right)^2 + \left(\frac{\pi}{4} - \frac{1}{16}\right)\right] \geq 0, \tag{27}$$

with $x^2 = N\pi^2\eta^2$. This constraint is always satisfied, hence the critical threshold for the mass of a stable extremal black hole is determined by the upper limit on the charge multiplicity: $N \lesssim 1/\eta^2 \approx 10^6 - 10^8$ for typical GUT scenarios. Therefore, the existence of stable, magnetically charged extremal black holes in our universe boils down to the viability of a black hole containing $N \gtrsim 10^6$ monopoles. The immediate problem with this is that we expect a period of inflation to dilute the monopole density to at most $\mathcal{O}(1)$ monopole per Hubble volume. Therefore, any process producing such a highly charged black hole would require a black hole to be present at or before inflation, which would most likely require the black hole to be formed via a tunnelling process.

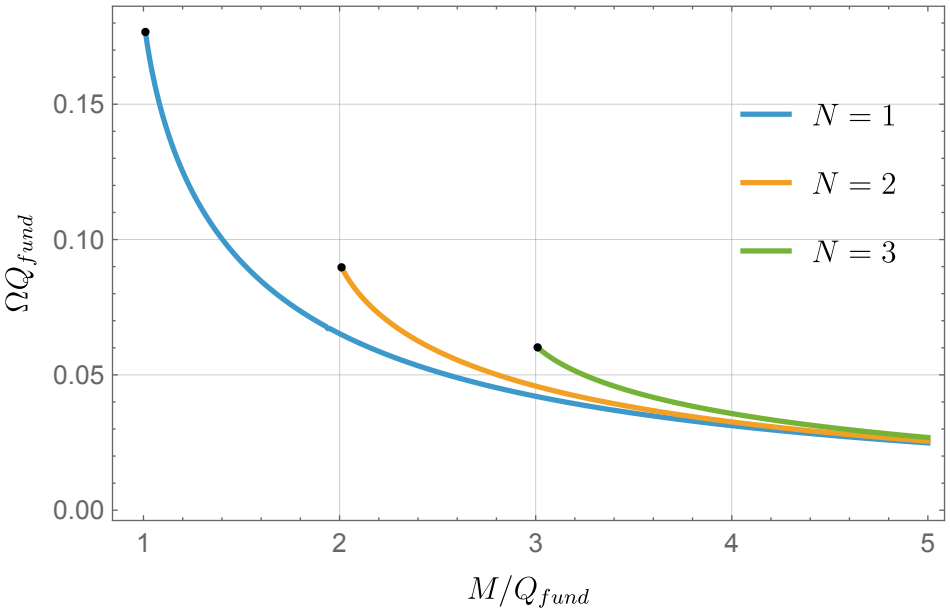

FIG. 1. *Instability of magnetically charged black holes.* Plot of Ω for a magnetically charged black hole as a function of its mass M, expressed in units of the fundamental magnetic charge. For reasons of dynamic range we only display results for low N. Each curve corresponds to a different winding number N, as indicated in the legend. The black dots mark the extremal limit: the minimum mass possible for a black hole with the given charge. Accordingly, the curves do not extend beyond this point. We adopt natural units with $G = c = \hbar = 4\pi\epsilon_0 = 1$, and set $\eta = 10^{-4}$ and $g = 1$, so that the fundamental magnetic charge is unity and the charge of the black hole is $Q = N$. The mass of the extremal black hole equals N.

Black hole pair creation was considered some time ago in the context of decay of cosmic strings [42–45] or magnetic fields [46] where the C-metric was used to construct a Euclidean instanton interpolating between the initial string or magnetic field to a pair of black holes accelerating apart. This type of nucleation would require some specific background that, while possible if cosmic strings are formed during the phase transition, will not produce black holes of sufficient size to accrete such a high monopole charge. The relevant process to consider is where a black hole spontaneously appears in the universe, as considered by Bousso and Hawking in [47]. Such a process reduced entropy, hence is strongly Boltzmann suppressed; for nucleation of an uncharged Nariai black hole this amplitude is:

$$\mathcal{P}_{BH} \sim e^{-\pi\ell^2/3G}, \tag{28}$$

where $\ell = \sqrt{\Lambda/3}$ is the de Sitter lengthscale of inflation. Since $\ell \sim 10^5$ Planck lengths, the nucleation rate of uncharged Nariai black holes is extremely low.

Bousso and Hawking computed the Boltzmann factor for the Nariai black hole as this geometry is regular – the general Schwarzschild de-Sitter solution has a mismatch of temperature between the black hole and cosmological horizons, thus a conical singularity in the Euclidean section. However, the presence of a conical singularity does not preclude a finite Euclidean action [48, 49], and finite action singular geometries contribute to tunnelling processes such as decay of the false vacuum [50, 51]. The relevant amplitude for spontaneous creation of a black hole near the start of inflation can be inferred from [52] where completely general Hawking-Moss instantons tunnelling from a lower to higher cosmological constant were considered with both seed and remnant black holes. Here, we are tunnelling from pure de Sitter to Schwarzschild de Sitter at the same cosmological constant. The semi-classical amplitude is therefore the exponent, $e^{-\mathcal{B}}$, of the entropy drop

$$\mathcal{B} = \mathcal{A}_{dS} - (\mathcal{A}_{bh} + \mathcal{A}_{cos}) \tag{29}$$

with $\mathcal{A}_{bh}$, $\mathcal{A}_{cos}$, $\mathcal{A}_{dS}$ the area of the black hole, the cosmological and the initial de Sitter horizons. This expression continuously interpolates from zero (no tunnelling) to Bousso-Hawking result $\mathcal{B}_{B-H} = \pi\ell^2/3$. Writing $\cos(3b) = 3\sqrt{3}M/\ell$ for the mass of the black hole, this action is

$$\mathcal{B} = \frac{\pi\ell^2}{3}\left(2\cos(2b) - 1\right) \tag{30}$$

Clearly, the nucleation of a small black hole is preferred over a larger one, however a small black hole will have a higher temperature thus be susceptible to evaporation, further, the black hole should have sufficient volume to absorb the high number of monopoles if we wish to create a stable extremal black hole. To get a ball-park estimate of this payoff, we note that the density of monopoles at production is determined by the Kibble mechanism [53], that predicts the formation of at most one magnetic monopole per correlation volume, with a domain size determined by the GUT phase transition scale. This yields a monopole number density of about 10^{-8} per Planck volume. Thus, for a nucleated black hole to capture $\sim 10^6$ monopoles, we require its volume to be $\sim 10^{14}$ Planck volumes. It can be shown that this limit on the black hole volume translates into the radius being approximately 1% of the Hubble scale. For small black hole radii, the instanton action with respect to the Bousso-Hawking one scales as $\mathcal{B}/\mathcal{B}_{B-H} \sim 6M/\ell = \mathcal{O}(10^{-2})$. Although the nucleation of this black hole is significantly more likely than the Bousso-Hawking nucleation, given that the actual value of this exponent is still 10^8 in Planck units, the amplitude for black hole formation remains extremely low, rendering the abundance of black holes nucleated via tunnelling negligible even when more general channels are included. On top of this, these black holes must form during the very early stages of inflation, so as a result, the subsequent e-folds of inflation would dilute their number density to an irrelevant level by today.

Finally, it is worth noting that the evidence for GUT's is indirect, and there is no experimental evidence for monopoles. However, upcoming experiments such as Hyper-Kamiokande [54] and DUNE [55] will significantly enhance sensitivity to proton decay, a key prediction of GUTs. Their results may soon provide decisive evidence for or against GUT-scale physics, with direct implications for the existence of magnetic monopoles (which are also the target of direct searches by the MoEDAL experiment [56]) and the associated black hole solutions.

IV. DISCUSSION

We have shown that charged extremal astrophysical black holes are inherently unstable once interactions with the environment and cosmological history are taken into account. For electric black holes, Schwinger pair production in the vicinity of a charged black hole must be weighted by computing not only the rate for a single electron-positron pair, but the integrated production rate within the spatial region where the electric field remains approximately of the same order of magnitude of its value at the event horizon. We considered a reasonable spatial volume for discharge set by the black hole radius, r_h^3, however the timescale for discharge we set to be the light crossing time for the black hole, which is incredibly short in astrophysical (let alone cosmological) terms. Once vacuum polarization becomes inhibited, the black hole can also discharge via ionization of neutral atoms in its vicinity. We focussed on Hydrogen to get a further bound on the likely discharge of an extremal black hole, which pushed up the lower bound on mass to $\mathcal{O}(10^{14}M_\odot)$. We again used the light crossing time for the black hole to obtain this conservative estimate. Considering that the light-crossing time for a $10^{14}M_\odot$ black hole is about 30 years – a blink of the eye in cosmological terms – this indicates our estimate may still be several orders of magnitude too low.

For extremal magnetic black holes that will not discharge via pair production, we computed the characteristic timescales associated with the Lee-Nair-Weinberg instability (an instability that arises when the monopole fields begin to leak out through the black hole horizon). We found that stable extremal magnetic black holes must have a charge of $10^6 - 10^8$ magnetic monopoles. Black holes with such large magnetic charges could only plausibly form before, or at the start of, inflation. We revisited the Bousso-Hawking amplitude for pair creation, including the possibility of singular geometries with finite action, and demonstrated that while the probability of spontaneously nucleating such black holes is strongly enhanced over the Bousso-Hawking rate, the requirement of large magnetic charge places a constraint on the mass of the nucleated black hole that renders the probability of formation still vanishingly low.

One might ask what happens in the case of dyonic black holes. If electric discharge is so efficient, but magnetic discharge suppressed – could a black hole with sufficient magnetic charge somehow stabilise? For an extremal dyonic black hole, the mass M is related to the magnetic P and electric Q charges via $M^2 = Q^2 + P^2$. The resulting electric field at the horizon is given by

$$\begin{aligned} |\mathbf{E}| =& \frac{c^4}{\sqrt{4\pi\epsilon_0 G^3 M^2}} \sqrt{(4\times 10^{60}\,[kg])\,\frac{M^2}{M_\odot^2} - \frac{\pi\hbar^2}{\mu_0 e^2 G}\,N^2} \\ \simeq& \frac{c^4}{\sqrt{4\pi\epsilon_0 G^3 M^2}} \sqrt{(4\times 10^{60}\,[kg])\,\frac{M^2}{M_\odot^2} - (2\times 10^{-14}\,[kg])\;N^2}. \end{aligned} \tag{31}$$

We observe that, due to the relative strength of the two terms under the square root, the inclusion of a magnetic charge does not significantly alter the conclusions drawn in Sec. II: the electric field remains strong enough in the supermassive regime to ionize hydrogen, and thus discharge still occurs.

We have focussed on charged black holes since they are often the most considered in high-energy and quantum-gravity studies. However, rotating black holes appears to contrast with the charged case as astrophysical black holes are observed to rotate very rapidly [57, 58]. Limits on spinning up a black hole via accretion disks were given by Thorne [59], who showed that a black hole can be spun up to a limiting angular momentum of $J = 0.998\,M^2$ once counteracting torque due to radiation emitted by the disk is taken into account. This has been revisited, including magnetic fields, by Mummery [60] showing a decrease of the limit to $J = 0.99\,M^2$. Both limits are of course below extremality. In addition, the angular momentum of a black hole can be extracted via the superradiance effect [61–64], or the amplification of low frequency waves scattering off the black hole thus reducing its rotational energy. Naturally, if there exist ultralight bosonic fields, currently considered as viable dark matter candidates [65], these can draw angular momentum from the black hole as form long-lived bosonic clouds around the black hole [62, 66, 67]. We also note a further, seemingly unavoidable, source of scattering that to the best of our knowledge has not yet been explored in the literature: the interaction of rotating black holes with the stochastic gravitational-wave background (for which there has recently been evidence via pulsar timing arrays in the nanohertz band [68]). Indeed, some frequencies of this background overlap with the superradiant window of any extremal Kerr black hole, since the angular velocity of an extremal Kerr horizon is

$$\Omega_h = \frac{c^3}{2GM} \sim 10^5 \frac{M_\odot}{M}\,[\mathrm{Hz}], \tag{32}$$

leading to a net, yet presumably very low, spin down of black holes.

In conclusion, the existence and possible observation of electrically or magnetically charged extremal black holes in our universe appears highly implausible. In contrast, while exact extremality may be ruled out, near-extremal rotating black holes remain viable astrophysical objects. The question remains whether they can be close enough to extremality to offer a promising setting for exploring quantum gravitational effects.

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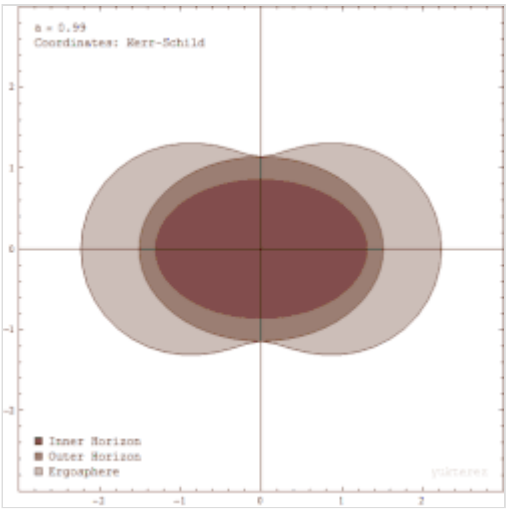
Extremal black hole

In theoretical physics, an **extremal black hole** is a black hole with the minimum possible mass that is compatible with its charge and angular momentum.[1] Extremal black holes have zero temperature. **Near-extremal black holes** with mass slightly above the extremal value have a simple horizon structure that make them valuable tools for black hole research.[2]

The concept of an extremal black hole is hypothetical, and none have thus far been observed in nature. However, they have proven useful as theoretical constructs.[3]



The ergosphere, outer horizon, and inner horizon of a Kerr black hole as its spin approaches extremal values (a=1).

Theorized characteristics

Theoretically, extremal black holes would have zero Hawking temperature and emit no Hawking radiation.[4][5] In supersymmetric theories, extremal black holes are often supersymmetric: they are invariant under several supercharges. This is a consequence of the BPS bound. Their black hole entropy[6] can be calculated in string theory.

One proposal, known as the "third law of black hole thermodynamics", says that no physical process can form a black hole with vanishing surface gravity.[7] This would disallow the formation of an extremal black hole; more specifically, no process involving a finite number of steps could produce a black hole without violating the weak energy condition.[8] A proof of this was published in 1986 by Werner Israel.[9] However, more recent work claims it contains an error and therefore extremal black holes are indeed possible.[10][11][3] The third law of thermodynamics for black holes has always been controversial.

Near-extremal black holes

A black hole whose mass is not far from the minimal possible mass that can be compatible with the given charges and angular momentum is known as a near-extremal black hole. Calculations regarding these bodies are usually performed using perturbation theory around the extremal black hole; the expansion parameter is called non-extremality. In supersymmetric theories, near-extremal black holes are often small perturbations of supersymmetric black holes. Such black holes have a very small Hawking temperature and consequently emit a small amount of Hawking radiation. Their entropy can often be calculated in string theory, much like in the case of extremal black holes, at least to the first order in non-extremality.

See also

- Black hole information paradox
- Quantum gravity
- Trans-Planckian problem
- Positive energy theorem

Notes

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External links

- Extremal RN Black Hole (https://web.archive.org/web/20120321162544/http://staff.science.uva.nl/~jpschaar/report/node35.html)
- *N*=2 extremal black holes (http://cdsweb.cern.ch/record/286302/files/PhysRevD.52.R5412.pdf)

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